MST207 Past Paper 2004

Part 1

Q1 Use separation of variables: $\int \frac{dy}{1+y^2} = \int x^2 dx \Rightarrow \tan^{-1}(y) = \frac{x^3}{3} + C \Rightarrow y = \tan\left(\frac{x^3}{3} + C\right)$

When x = 0, y = 1 then tan(C) = 1 and so C = π/4 So : $y = \tan\left(\frac{x^3}{3} + \frac{\pi}{4}\right)$

Q2 a

$$\overrightarrow{OD} = \frac{\mathbf{0} + (\boldsymbol{i} + 0 + 0) + \left(\frac{\boldsymbol{i}}{2} + \frac{\sqrt{3}}{2}\boldsymbol{j} + 0\right)}{3} = \frac{\left(\frac{3}{2}\boldsymbol{i} + \frac{\sqrt{3}}{2}\boldsymbol{j}\right)}{3} = \left(\frac{1}{2}\boldsymbol{i} + \frac{1}{2\sqrt{3}}\boldsymbol{j}\right)$$

b

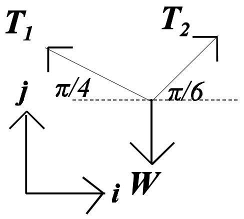
$$\overrightarrow{OC} = \overrightarrow{OD} + \lambda\boldsymbol{k} = \left(\frac{1}{2}\boldsymbol{i} + \frac{1}{2\sqrt{3}}\boldsymbol{j} + \lambda\boldsymbol{k}\right) \Rightarrow |\overrightarrow{OC}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2\sqrt{3}}\right)^2 + \lambda^2} = 1$$

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2\sqrt{3}}\right)^2 + \lambda^2 = 1 \Rightarrow \lambda = \sqrt{\frac{2}{3}} \Rightarrow |\overrightarrow{OC}| = \left(\frac{1}{2}\boldsymbol{i} + \frac{1}{2\sqrt{3}}\boldsymbol{j} + \sqrt{\frac{2}{3}}\boldsymbol{k}\right)$$

c

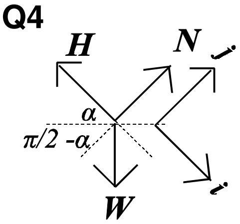
$$\cos(\theta) = \frac{\overrightarrow{OC} \cdot \overrightarrow{OD}}{|\overrightarrow{OC}|\,|\overrightarrow{OD}|} = \frac{\frac{1}{3}}{\frac{1}{\sqrt{3}}} = \frac{1}{\sqrt{3}}$$

Q3



W = -|**W**|**j** = -mg**j**
$\mathbf{T_1}$ = -|$\mathbf{T_1}$|cos(π/4)**i** +|$\mathbf{T_1}$|sin(π/4)**j** = -(|$\mathbf{T_1}$|/√2)**i** + (|$\mathbf{T_1}$|/√2)**j**
$\mathbf{T_2}$ = |$\mathbf{T_2}$|cos(π/6)**i** +|$\mathbf{T_2}$|sin(π/6)**j** = (√3|$\mathbf{T_2}$|/2)**i** +(|$\mathbf{T_2}$|/2)**j**
In equilibrium **W** + $\mathbf{T_1}$ + $\mathbf{T_2}$ = 0
Resolving in **j**-direction: -mg**j** + (|$\mathbf{T_1}$|/√2)**j** + (|$\mathbf{T_2}$|/2)**j** = 0
So: mg = |$\mathbf{T_1}$|/√2 + |$\mathbf{T_2}$|/2
Resolving in **i**-direction: -|$\mathbf{T_1}$|/√2 **i** = g**i** so |$\mathbf{T_1}$| = √2 g; -(|$\mathbf{T_1}$|/√2)**i** + (√3|$\mathbf{T_2}$|/2)**i** = 0
So: |$\mathbf{T_1}$|/√2 = √3|$\mathbf{T_2}$|/2 and so |$\mathbf{T_1}$| = √3|$\mathbf{T_2}$|/√2
Substituting: mg = √3|$\mathbf{T_2}$|/(√2)² + |$\mathbf{T_2}$|/2 = ½(1 + √3)|$\mathbf{T_2}$| So:|$\mathbf{T_2}$| = 2mg/(1 + √3) as required.
Substituting this gives: |$\mathbf{T_1}$| = (√6 mg)/(1 + √3) as required.

Q4



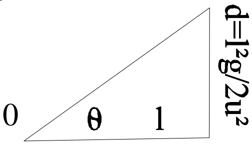
H = -k(x - l_0)**ŝ** = -k(x - l_0)**i** since **ŝ** = **i**
N = |**N**|**j**
W = |**W**|sin(α)**i** -|**W**|cos(α)**j** = mg sin(α)**i** - mg cos(α)**j**
$\ddot{\mathbf{x}} = \ddot{x}\mathbf{i}$
Since we have motion then: m$\ddot{\mathbf{x}}$ = **H** + **N** + **W**
Resolving in **j**-direction gives: 0 = |**N**|**j** - |**W**|cos(α)**j**
So:|**N**| = |**W**|cos(α) = mg cos(α)
Resolving in **i**-direction gives: m$\ddot{x}$**i** = -k(x - l_0)**i** + mg sin(α)**i**
m$\ddot{x}$ = -k(x - l_0) + mg sin(α)
So equation of motion is: $\ddot{x}$ + kx/m = g sin(α) + kl_0/m
Equilibrium position, x = l_{eq}, is when mass is stationary; i.e. when $\ddot{x}$ = 0. Substituting gives: l_{eq} = (mg sin(α) + kl_0)/k

Q5 a Initially θ = 0. Since the trajectory is: $y = h + x\tan\theta - \frac{gx^2}{2u^2}\left(1 + \tan^2\theta\right)$
Simplifying gives:

$$y = h + 0 - \frac{l^2 g}{2u^2}(1 + 0) \text{ or } y = h - \frac{l^2 g}{2u^2}$$

Since the target is at the same height as the launch then vertical distance that the shot misses the target is: $d = \frac{l^2 g}{2u^2}$

b This time the shot is aimed at a height d above the target. So the tangent of the angle of projection is: $\tan\theta = \frac{lg}{2u^2}$ Substituting in the trajectory equation gives:



$$y = h + l\times\frac{lg}{2u^2} - \frac{gl^2}{2u^2}\left(1 + \left(\frac{lg}{2u^2}\right)^2\right) \Rightarrow y = h - \frac{l^4 g^3}{8u^6} \quad \text{and} \quad \frac{l^4 g^3}{8u^6} \neq 0$$

It will only hit the target if the trajectory, y, equals h at l. Since the expression is not zero then the object will fall below the target. We know the expression is not zero since it is composed of all positive elements.

Q6 $A = 8m^2$; $\Theta_{in} = 19°C$; $\Theta_{out} = 4\ °C$; $b = 0.1m$; $k = 0.6Wm^{-1}K^{-1}$; $h_{in} = 10Wm^{-2}K^{-1}$ $h_{out} = 150Wm^{-2}K^{-1}$

a $$q = \left(\frac{1}{h_{in}} + \frac{1}{h_{out}} + \frac{b}{k}\right)^{-1} A\left(\Theta_{in} - \Theta_{out}\right) \Rightarrow q \approx 3.66\times8\times15 \approx 439.02\text{W}$$

b $$q = h_{in}A\left(\Theta_{in} - \Theta_1\right) \Rightarrow \Theta_1 = \Theta_{in} - \frac{q}{Ah_{in}} = 19 - \frac{439.02}{10\times8} \approx 13.5°\text{C}$$

Q7

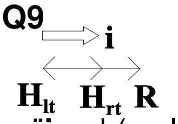
$$\begin{matrix} \boldsymbol{R_1} \\ \boldsymbol{R_2} \\ \boldsymbol{R_3} \end{matrix}\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 2 & 1 & a \\ 2 & 0 & b & 4 \end{array}\right] \qquad \begin{matrix} \boldsymbol{R_1} \\ \boldsymbol{R_{2a}}=\boldsymbol{R_2}-2\boldsymbol{R_1} \\ \boldsymbol{R_{3a}}=\boldsymbol{R_3}-2\boldsymbol{R_1} \end{matrix}\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -2 & -1 & a-4 \\ 0 & -4 & b-2 & 0 \end{array}\right] \qquad \begin{matrix} \boldsymbol{R_1} \\ \boldsymbol{R_{2a}} \\ \boldsymbol{R_{3b}}=\boldsymbol{R_{3a}}-2\boldsymbol{R_{2a}} \end{matrix}\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -2 & -1 & a-4 \\ 0 & 0 & b & -2(a-4) \end{array}\right]$$

For there to be infinitely many solutions then all the entries on row R_{3b} have to be zero.
Hence: b = 0 and: -2(a - 4) = 0. So 2a = 8 So: a = 4

Q8 a $$\begin{bmatrix} 10 & 2 & 0 \\ 4 & 11 & 6 \\ -2 & -1 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \\ -12 \end{bmatrix} = 12\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$ So corresponding eigenvalue is 12.

b
Let $\lambda = 6$; $(10 - 6)x + 2y + 0 = 2$; $4x + (11 - 6)y + 6z = 0$; $-2x - y + (6 - 6)z = 0$
The first and third equations give $-2x = y$. So if x = 1 the eigenvector so far is $[1\ -2\ z]^T$
Substituting these into equation 2 gives z = 1. So corresponding eigen vector is $[1\ -2\ 1]^T$

Q9



$\mathbf{H}_{lt} = -k(x - l_0)\mathbf{i}$
$\mathbf{H}_{rt} = -2k(d - x - l_0)(-\mathbf{i}) = 2k(d - x - l_0)\mathbf{i}$
$\mathbf{R} = -r\dot{x}\mathbf{i}$
$m\ddot{x}\mathbf{i} = -k(x - l_0)\mathbf{i} + 2k(d - x - l_0)\mathbf{i} - r\dot{x}\mathbf{i}$ which results in: $m\ddot{x} + r\dot{x} + 3kx = k(2d - l_0)$, eqn of motion.

Q10 a By principle of Conservation of momentum: $m(u\mathbf{i} + v\mathbf{j}) + m\mathbf{0} = mp\mathbf{i} + mq\mathbf{j}$. Resolving into **i**- and **j**-directions gives p = u and q = v and so after collision one particle has velocity u**i** and the other v**j**.
b If elastic then K.E. before = K.E. after. K.E. before = $\frac{1}{2}mu^2$ where $u = |\mathbf{u}| = \sqrt{(u^2 + v^2)}$ so K.E. before = $\frac{1}{2}m(u^2 + v^2)$.
K.E. after = $\frac{1}{2}mp^2 + \frac{1}{2}mq^2$ where $p = |\mathbf{p}| = u$ and $q = |\mathbf{q}| = v$ so K.E. after = $\frac{1}{2}m(u^2 + v^2)$.
Hence collision is elastic as K.E. before = K.E. after.

Q11 a $f_x = 8xy + y^2 + 12$ $f_y = 4x^2 + 2xy = x(4x + 2y)$
Stationary points are found by solving the simultaneous equations $f_x =$ and $f_y = 0$. i.e.
$8xy + y^2 + 12 = 0$ and $x(4x + 2y) = 0$
The second one gives a solution of $x = 0$. Substituting this into first equation gives $y^2 = -12$ which has solutions $\pm 2\sqrt{3}i$ and so $(0, +2\sqrt{3}i)$ and $(0, -2\sqrt{3}i)$ are two stationary points.
Now: $(4x + 2y) = 0$ gives: $-2x = y$. Substituting into $f_x = 0$ gives $x = \pm 1$. When $x = +1$, $y = -2$ and when $x = -1$, $y = +2$ (found by substituting values of x into $f_y = 0$
So all stationary points are: $(0, +2\sqrt{3}i)$; $(0, -2\sqrt{3}i)$; $(1, -2)$; $(-1, 2)$
b $A = f_{xx} = 8y$ $C = f_{yy} = 2x$ $B = f_{xy} = 8x + 2y$
With point where x is positive: $A = f_{xx}(1,-2) = -16$; $C = f_{yy}(1,-2) = 2$; $B = f_{xy}(1,-2) = 8 - 4 = 4$
$AC - B^2$ at $(1,-2) = -32 + 16 = -16 < 0$ hence stationary point is a Saddle Point.

Q12 i Plot Y_N against h^4
ii $C \approx (Y_2 - Y_1)/(h_1^4 - h_2^4) = (0.370098 - 0.368018)/((0.25)^4 - (0.125)^4) \approx 0.568$
For 15 d.p. accuracy: $h^4 < 0.5\times10^{-15}/C$ Thus $h < 0.000172$

Q13 To find the area we let f, the surface density function, = 1. So:

$$A = \int_{x=0}^{x=\pi/2}\int_{y=1-2x/\pi}^{y=\cos x} 1\,dydx + \int_{x=\pi/2}^{x=\pi}\int_{y=\cos x}^{y=1-2x/\pi} 1\,dydx = \int_{x=0}^{x=\pi/2}\left(\cos x - 1 - \frac{2x}{\pi}\right)dx + \int_{x=\pi/2}^{x=\pi}\left(1 - \frac{2x}{\pi} - \cos x\right)dx$$

$$= \left[\sin x - x - \frac{x^2}{\pi}\right]_0^{\pi/2} + \left[x - \frac{x^2}{\pi} - \sin x\right]_{\pi/2}^{\pi} = 2 - \frac{\pi}{2}$$

Q14 Since rotating about fixed axis then $K.E = \frac{1}{2}I\omega^2 = 100$ J
$I = I_{rod} + 2I_{disk} = \frac{1}{2}m_{rod}r_{rod}^2 + M_{disk}R_{disk}^2$.
So:

$$\omega^2 = \frac{100}{\frac{1}{2}(1)(0.02)^2 + 3(0.25)^2} \approx 532.765$$

$$\omega = \sqrt{\omega^2} \approx 23.08 \text{ rad s}^{-1}$$

Part 2

Q15 i Characteristic equations is: $4\lambda^2 + 4\lambda + 5 = 0$ which has solutions $-½ \pm i$ and so homogeneous solution is: $x = e^{-½t}(C\cos(t) + D\sin(t))$

Let: $x_p = ke^{-½t}$; then: $x' = -½ke^{-½t}$ and: $x'' = ¼ke^{-½t}$

Substituting gives k = 1, so solutions so far is: $x = e^{-½t}(C\cos(t) + D\sin(t)) + e^{-½t}$

Let: $x_p = at^2 + bt + c$; then: $x' = 2at + b$; and: $x'' = 2a$

Substituting gives a = 1, b = 0 and c = 1 so solutions so far is:

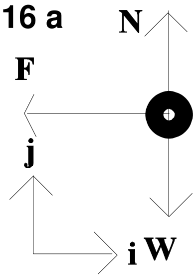
$x = e^{-½t}(C\cos(t) + D\sin(t)) + e^{-½t} + t^2 + 1$

Since x(0) = 0, substituting gives: 0 = C + 1 + 1 so C = -2. So the solution so far is:

$x = e^{-½t}(D\sin(t) - 2\cos(t)) + e^{-½t} + t^2 + 1$

Since $x'(0) = 0$ then $x' = e^{-½t}(D\cos(t) + 2\sin(t) - ½(D\sin(t) - 2\cos(t) + 1)) + 2t$ which, on substitution gives $D = -½$. So the particular solution is: $x = e^{-½t}(1 - 2\cos(t) - ½\sin(t)) + t^2 + 1$

Q16 a



$\ddot{\mathbf{x}} = \ddot{x}\mathbf{i}$; $\mathbf{v} = v\mathbf{i}$; $\mathbf{N} = |\mathbf{N}|\mathbf{j}$; $\mathbf{F} = -|\mathbf{F}|\mathbf{i}$; $|\mathbf{F}| = \mu|\mathbf{N}|$; $\mathbf{W} = -|\mathbf{W}|\mathbf{j} = -mg\mathbf{j}$

Since there is motion we have: $\mathbf{N} + \mathbf{F} + \mathbf{W} = m\ddot{\mathbf{x}}$

Resolving in **j**-direction gives: $|\mathbf{N}| = mg$. Consequently $|\mathbf{F}| = \mu mg$

Since we have motion then by Newton's Second Law, resolving in **i**-direction gives:

$m\ddot{x}\mathbf{i} = -|\mathbf{F}|\mathbf{i} = -\mu mg\mathbf{i}$. This simplifies to: $\ddot{x} = -\mu g$

$$\ddot{x} = v\frac{dv}{dx} = -\mu g \Rightarrow \int v\,dv = -\mu g\int dx \Rightarrow \frac{v^2}{2} = -\mu g x + C$$

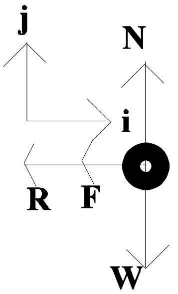
Initially when x = 0, $v = u_0$ So:

$$\frac{u_0^2}{2} = C \Rightarrow \frac{v^2}{2} = \frac{u_0^2}{2} - \mu g x \Rightarrow v^2 = u_0^2 - 2\mu g x \Rightarrow v = \sqrt{u_0^2 - 2\mu g x} \quad \text{or: } x = \frac{u_0^2 - v^2}{2\mu g}$$

When has come to rest, v = 0. Substituting this, and all the other given values for the constants gives x as:

$$x = \frac{u_0^2 - v^2}{2\mu g} = \frac{2^2 - 0}{2\times 0.02\times 9.81} \approx 10.19\text{m (to 2.d.p.)}$$

b



$\mathbf{R} = -|\mathbf{R}|\mathbf{i} = -kv^2\mathbf{i}$

All other forces and vectors defined as above and $|\mathbf{F}| = \mu mg$ as before.

Since there is motion we have: $\mathbf{N} + \mathbf{F} + \mathbf{R} + \mathbf{W} = m\ddot{\mathbf{x}}$

Resolving in **i**-direction now gives: $m\ddot{x}\mathbf{i} = -|\mathbf{F}|\mathbf{i} - |\mathbf{R}|\mathbf{i} = -\mu mg\mathbf{i} - kv^2\mathbf{i}$

Hence: $\ddot{x} = -\mu g - kv^2/m$

$k = c_2D^2 = 0.2 \times (0.05)^2 = 0.0005$ and $k/m = 0.0005/2 = 0.00025 = 1/4000$

$$\ddot{x} = v\frac{dv}{dx} = -\mu g - \frac{v^2}{4000} \Rightarrow \int \frac{v\,dv}{\mu g + \frac{v^2}{4000}} = 2000\int \frac{\frac{v\,dv}{2000}}{\frac{4000\mu g + v^2}{4000}} = 2000\int \frac{2v\,dv}{4000\mu g + v^2} = -\int dx$$

$$2000\ln\left(4000\mu g + v^2\right) = -x + C \qquad \text{(integrate by substitution)}$$

Initially, x = 0 and $v = u_0$. So:

$$c = 2000\ln\left(4000\mu g + u_0^2\right) \text{ so } x = 2000\left(\ln(4000\mu g + u_0^2) - \ln(4000\mu g + v^2)\right)$$

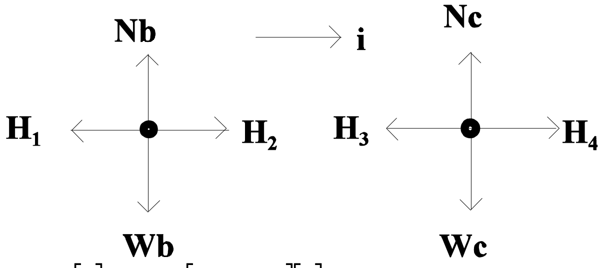
$$x = 2000\ln\left(\frac{4000\mu g + u_0^2}{4000\mu g + v^2}\right)$$

When comes to rest v = 0. Substituting this and all the other given constants gives:

$$x = 2000\ln\left(\frac{4000\times 0.02\times 9.81 + 2^2}{4000\times 0.02\times 9.81 + 0}\right) = 2000\ln(1.0051) \approx 10.17\text{m (to 2.d.p.)}$$

c Adding air resistance stops the mass about 2cm sooner. Two cm over approximately 10.2m represents about 0.2%. So the effect of air resistance accounts for about 0.2% of the resistive force and friction for 99.8% of the resistive force. Air resistance is effectively negligible in this case.

Q17 a



$\Delta \mathbf{H}_1 = -kx\mathbf{i}$
$\Delta \mathbf{H}_2 = 4k(y - x)\mathbf{i}$
$\Delta \mathbf{H}_3 = -\Delta \mathbf{H}_2$
$\Delta \mathbf{H}_4 = -ky\mathbf{i}$
$m\ddot{x}\mathbf{i} = \Delta \mathbf{H}_1 + \Delta \mathbf{H}_2 = -5kx\mathbf{i} + 4ky\mathbf{i}$
So: $\ddot{x} = k/m(-5x + 4y)$
Sim: $\ddot{y} = k/m(4x - 5y)$

So: $$\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = \frac{k}{m}\begin{bmatrix}-5 & 4\\ 4 & -5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

b Eigenvalues of the matrix are -1 and -9 and hence of the whole system are -k/m and -9k/m. The normal mode angular frequencies are given by $\sqrt{(-\lambda)}$, where λ is an eigenvalue of the system. So the normal mode angular frequencies are $\sqrt{k}/\sqrt{m}$ and $3\sqrt{k}/\sqrt{m}$
c Eigenvectors are [1,1] and [-1,1] respectively. Hence general solution is:

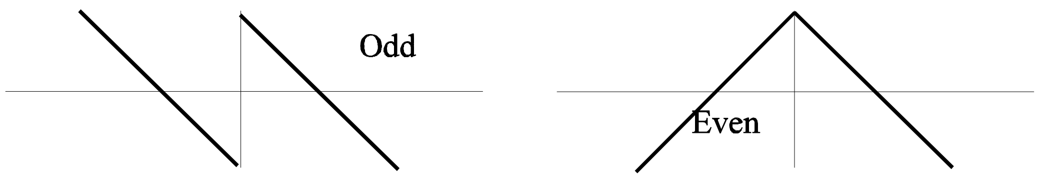
$$\begin{bmatrix}x(t)\\ y(t)\end{bmatrix} = C_1\begin{bmatrix}1\\ 1\end{bmatrix}\cos\left(\sqrt{\frac{k}{m}}t + \varphi\right) + C_2\begin{bmatrix}-1\\ 1\end{bmatrix}\cos\left(3\sqrt{\frac{k}{m}}t + \varphi\right)$$

d All phase angles, φ, are zero. At t = 0, then x(0) = 0.1m and y(0) = 0.2m. Substituting gives C1 = 0.15m (15 cm) and C2 = 0.05m (5cm). Hence at time t, the positions x(t) and y(t) are given (in metres) by: $$\begin{bmatrix}x(t)\\ y(t)\end{bmatrix} = 0.15\begin{bmatrix}1\\ 1\end{bmatrix}\cos\left(\sqrt{\frac{k}{m}}t\right) + 0.05\begin{bmatrix}-1\\ 1\end{bmatrix}\cos\left(3\sqrt{\frac{k}{m}}t\right)$$

Q18. This is virtually identical to the 2001 paper Q18

Q 19 a

$$f_{odd}(t) = \begin{cases} \tfrac{1}{2} - t & 0 \le t \le 1 \\ -(\tfrac{1}{2} - t) & -1 \le t \le 0 \end{cases} \qquad f_{even}(t) = \begin{cases} \tfrac{1}{2} - t & 0 \le t \le 1 \\ \tfrac{1}{2} + t & -1 \le t \le 0 \end{cases}$$



b

$$A_0 = \int_0^1 (\tfrac{1}{2} - t)dt = \left[\tfrac{1}{2}t - \tfrac{1}{2}t^2\right]_0^1 = 0$$

$$A_r = 2\int_0^1 (\tfrac{1}{2} - t)\cos(r\pi t)dt = \frac{2}{r^2\pi^2}\left[-\cos(r\pi t)\right]_0^1 = \frac{2}{r^2\pi^2}(1 - \cos(r\pi)) \qquad (r = 1, 2, \ldots)$$

c

$$B_r = 2\int_0^1 (\tfrac{1}{2} - t)\sin(r\pi t)dt = \frac{1 + \cos(r\pi)}{r\pi}$$

(A_r and B_r by integration by parts)

d The cosine function would be better because the terms decrease by a factor of $1/r^2$ compared to the sine function which decreases by a factor of only 1/r each time.

Q 20 a Path is a circle centred on the origin with radius = 1
b $\dot{\mathbf{r}}(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j} + 0\mathbf{k}$ and $\mathbf{F}(\mathbf{r}) = \alpha^2\sin(t)\mathbf{i} + \cos(t)\mathbf{j} + \sin(t)\cos(t)\mathbf{k}$

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{r})\cdot\dot{\boldsymbol{r}}(t)\,dt = \pi(1-\alpha^2)$$

c If the field is conservative then the line integral of any closed loop is zero. The above is a line integral of a closed loop (circle). This is zero when $\alpha^2 = 1$ hence when $\alpha = \pm 1$. It is not conservative for all other values of α.
d **curl F** = $(1 - \alpha^2)\mathbf{k}$ and so **F** is conservative only when $\alpha = \pm 1$.