MST207 Past Paper 2003

Part 1

Q1 Use integrating factor method: $g(x) = -\dfrac{1}{x} \Rightarrow p = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \dfrac{1}{x}$

$$\int\left(\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2}\right) = \int x dx \Rightarrow \frac{y}{x} = \frac{x^2}{2} + C \Rightarrow y = \frac{x^3}{2} + Cx$$

When x = 1, y = 3/2 then C = 1 So : $y = \dfrac{x^3}{2} + x$

Q2 i $|\boldsymbol{a}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$ $\qquad |\boldsymbol{b}| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{24} = 2\sqrt{6}$

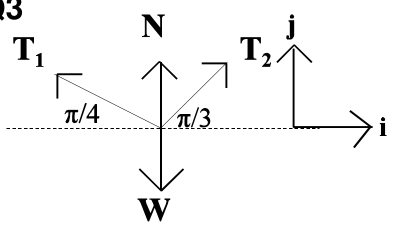
ii $\cos\theta = \dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \dfrac{1}{12}(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})\cdot(-2\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}) = \dfrac{4}{12} = \dfrac{1}{3}$

iii

$$\boldsymbol{c} = \boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & -1 \\ -2 & 4 & 2 \end{vmatrix} = (8\boldsymbol{i} + 0\boldsymbol{j} + 8\boldsymbol{k}) \quad \text{so } \hat{\boldsymbol{c}} = \frac{1}{\sqrt{2}}(\boldsymbol{i} + \boldsymbol{k})$$

Q3

i



W = -|**W**|**j** = -mg**j**

$\mathbf{T_1}$ = -|$\mathbf{T_1}$|cos(π/4)**i** +|$\mathbf{T_1}$|sin(π/4)**j** = -|$\mathbf{T_1}$|/√2 **i** + |$\mathbf{T_1}$|/√2 **j**

$\mathbf{T_2}$ = |$\mathbf{T_2}$|cos(π/3)**i** +|$\mathbf{T_2}$|sin(π/3)**j** = g**i** + √3g**j**

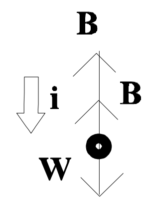
N = |**N**|**j**

In equilibrium **W** + $\mathbf{T_1}$ + **N** + $\mathbf{T_2}$ = 0

Resolving in **i**-direction: -|$\mathbf{T_1}$|/√2 **i** = g**i** so |$\mathbf{T_1}$| = √2 g

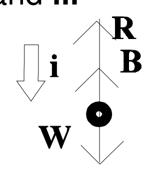
ii Resolving in **j**-direction: -mg**j** + |$\mathbf{T_1}$|/√2 **j** + √3g**j** + |**N**|**j** = 0
Now m just ceases to be in contact with the floor when |**N**| = 0 so the minimum value of m is when |**N**| > 0 So: g + √3g = mg when block just about to lift from floor so m bust be just bigger than this, ie: m > 1 + √3, as required.

Q4 i



B = -|**B**|**i** (the force due to one balloon)
W = |**W**|**i** = mg**i** = g**i** since m = 1kg
Since we are in equilibrium: **B** + **B** + **W** = 0 = -2|**B**|**i** + mg**i** and so |**B**| = ½g and so the force exerted by one balloon is **B** = -½g**i** and the magnitude of the force of one balloon = ½g

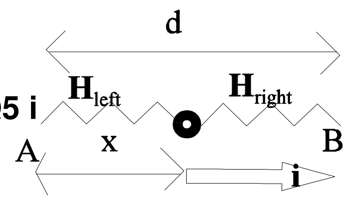
ii and **iii**



R = $-c_2D^2v^2$**i**
$\ddot{\mathbf{x}} = \ddot{x}\mathbf{i}$
Since there is motion: m$\ddot{\mathbf{x}}$ = **W** + **B** + **R**
But at terminal velocity $\ddot{x}$=0 = mg**i** - ½g**i** - $c_2D^2v^2$**i**
So: 0 = ½g - 0.2v^2

So: $v_{term} = \sqrt{\dfrac{5g}{2}} \approx 4.952$ So the terminal velocity is approx 5ms^{-1} (to 1 s.f.)
(I am grateful to Susan Meeks for her simpler approach to solving this part)

Q5 i



$\mathbf{H}_{\mathbf{left}} = -k_{left}(x - l_0)\hat{\mathbf{s}} = -k(x - l_0)\mathbf{i}$ $\quad (\hat{\mathbf{s}} = \mathbf{i})$
$\mathbf{H}_{\mathbf{right}} = -k_{right}(d - x - l_0)\hat{\mathbf{s}} = -2k(d - x - l_0)(-\mathbf{i})$ $(\hat{\mathbf{s}} = -\mathbf{i})$ $= 2k(d - x - l_0)\mathbf{i}$
Where x is the equilibrium position of the mass measured from A.

ii Because no motion: $\mathbf{H}_{\text{left}} + \mathbf{H}_{\text{right}} = 0 = -k(x - l_0)\mathbf{i} + 2k(d - x - l_0)\mathbf{i}$
$(x - l_0) = 2(d - x - l_0)$ So $3x = 2d - l_0$ and so $x = ⅓(2d - l_0)$
So the equilibrium position measured form the left is $l_{eq} = ⅓(2d - l_0)$ metres.
iii By Newton's Second Law since we now have motion: $m\ddot{\mathbf{x}} = \mathbf{H}_{\text{left}} + \mathbf{H}_{\text{right}}$
So: $m\ddot{x}\mathbf{i} = -k(x - l_0)\mathbf{i} + 2k(d - x - l_0)\mathbf{i}$ hence $m\ddot{x} + 3kx = 2kd - kl_0$
(I am grateful to Susan Meeks for her corrections here)

$$\ddot{x} + \frac{3k}{m}x = \frac{2kd - kl_0}{m} \Rightarrow \ddot{x} + \frac{3k}{m}x = \frac{3k}{m}l_{eq} \Rightarrow \text{Let: } \omega^2 = \frac{3k}{m} \text{ So } x(t) = l_{eq} + B\cos(\omega t) + C\sin(\omega t)$$

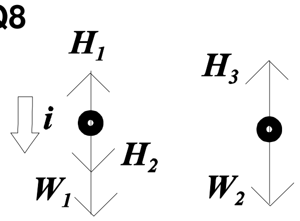
(HB 1 page 33)

Initially at t = 0, displacement, $x(0) = ½d$ and velocity $x'(0) = 0$
So: $½d = l_{eq} + B\cos(0) + C\sin(0)$ and so $B = ½d - l_{eq} = ⅓(l_0 - ½d)$ and: $0 = -B\omega\sin(0) + C\omega\cos(0)$ and so $C = 0$
So: $x(t) = ⅓(l_0 - ½d)\cos(\omega t)$ and so the amplitude is $⅓(l_0 - ½d)$.

Q6 Matrix reduces to: $\left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & -1 & -4 & 6 \\ 0 & 0 & -6 & 12 \end{array}\right]$ This has a unique solution which is: $x_3 = -2$; $x_2 = 2$; $x_1 = 1$

Q7 Eigenvalues are +2 and -2, hence Unstable Saddle. Corresponding eigenvectors are [1,1] and [-1,1]. See Handbook/Unit for relevant sketch.

Q8



$\mathbf{W_1} = mg\mathbf{i}$ $\mathbf{W_2} = 2mg\mathbf{i}$ $\mathbf{H_1} = -k(x - l_0)\mathbf{i}$
$\mathbf{H_2} = +2k(y - x - 2l_0)\mathbf{i}$ and $\mathbf{H_3} = -\mathbf{H_2}$
Static so: $\mathbf{0} = \mathbf{W_1} + \mathbf{W_2} + \mathbf{H_1} + \mathbf{H_2} + \mathbf{H_3} = \mathbf{W_1} + \mathbf{W_2} + \mathbf{H_1}$
Giving: $3mg = kx - kl_0$ and so $x = 3mg/k + l_0$.
Now $\mathbf{0} = \mathbf{W_2} + \mathbf{H_3}$ so: $2mg = 2k(y - x - 2l_0)$ and so $y = 4mg/k + 3l_0$

Q9 Conservation of momentum: $m_1u\mathbf{i} + m_2\mathbf{0} = m_1v_1\mathbf{i} + m_2v_2\mathbf{i}$ (because all in same direction is effectively one dimensional). Using Newton's Law of restitution we know that e = 1 (elastic), so $v_2\mathbf{i} - v_1\mathbf{i} = -e(\mathbf{0} - u\mathbf{i})$ and so $v_2 - v_1 = u$. Hence: $m_1u = m_1v_1 + m_2(u + v_1)$ which on rearranging gives $v_1 = u(m_1 - m_2)/(m_1+m_2)$. (Can do this using Conservation of momentum and equating K.E. before and after but is not as straight forward.)

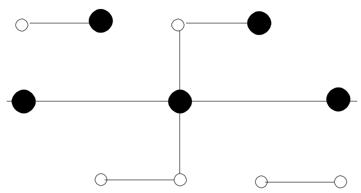
Q10 i Linear momentum, $\mathbf{p} = m\dot{\mathbf{r}} = -m\omega\sin(\omega t)\mathbf{i} + m\omega\cos(\omega t)\mathbf{j}$.
Angular momentum $\mathbf{l} = \mathbf{r} \times \mathbf{p} = m\omega\mathbf{i} - m\omega\cos(\omega t)\mathbf{j} - m\omega\sin(\omega t)\mathbf{k}$
ii Ext force, $\mathbf{F} = \dot{\mathbf{p}} = -m\omega^2\cos(\omega t)\mathbf{j} - m\omega^2\sin(\omega t)\mathbf{k}$
iii For torque law to be true $\mathbf{l}' = \mathbf{\Gamma} = \mathbf{r} \times \mathbf{F}$
This is true since they both $= m\omega^2\sin(\omega t)\mathbf{j} - m\omega^2\cos(\omega t)\mathbf{k}$

Q11 $f_x = 4x - 4y - 4$; $f_y = -4x - 6y + 4$
The 2 simultaneous equations $4x - 4y - 4 = 0$ and $-4x - 6y + 4 = 0$ have one solution of $x = 1$ and $y = 0$. So the point (1,0) is a stationary point. $A = f_{xx} = 4$; $C = f_{yy} = -6$; $B = f_{xy} = -4$
$AC - B^2$ at $(1,0) = -24 - 16 = -40 < 0$ hence stationary point is a Saddle Point.

Q12 i $x_0 = 0$ $Y_0 = y_0 = 1$ $h = 0.1$ $r = 0$
$F_{1,0} = f(x_0,Y_0) = 1 + 0 + 1 = 2$
$F_{2,0} = f(x_0 + ½h, Y_0 + ½hF_{1,0}) = f(0.05, 1.1) = 2.15$
$F_{3,0} = f(x_0 + ½h, Y_0 + ½hF_{2,0}) = f(0.05, 2.1575) = 2.1575$
$F_{4,0} = f(x_0 + h, Y_0 + hF_{3,0}) = f(0.1, 1.21575) = 2.31575$

$Y_1 = Y_0 + h(F_{1,0} + 2F_{2,0} + 2F_{3,0} + F_{4,0})/6 = 1.216$ to 3 d.p.
ii Order of local truncations error is $O(h^5)$ i.e. is of the order of $h^5 = 10^{-5}$

Q13 i



ii Coefficients of the cosine terms will be zero because the function is an odd function and with odd functions it is the sine functions that are used to approximate them, the cosine functions being essentially redundant.

iii The first non-zero term in the Fourier series for an odd function is B_1. This function has period = 2. So:

$$B_r = \frac{4}{\tau}\int_0^{\frac{\tau}{2}} f(t)\sin\left(\frac{2r\pi t}{\tau}\right)dt = 2\int_0^1 (1)\sin(r\pi t)dt$$

$$B_1 = 2\int_0^1 \sin(\pi t)dt = \frac{2}{\pi}[-\cos(\pi t)]_0^1 = \frac{2}{\pi}(-\cos(\pi)-(-\cos 0)) = \frac{2}{\pi}(2) = \frac{4}{\pi}$$

$$\text{First non zero term} = \frac{4}{\pi}\sin(\pi t)$$

Q14 Upper x limit is when $x = 2x - x^2$ which is only when $x = 1$

$$A = \int_0^1\int_{y=x}^{y=2x-x^2}(x^2)dydx = \int_0^1 x^2(2x - x^2 - x)dx = \int_0^1 (x^3 - x^4)dx = \frac{1}{20}$$

Part 2

Q15 i Characteristic equations is: $\lambda^2 + 2\lambda + 10 = 0$ which has solutions $-1 \pm 3i$ and so homogeneous solution is: $x = e^{-t}(C\cos 3t + D\sin 3t)$
Let: $x_p = a\cos\Omega t + b\sin\Omega t$; then: $x' = -a\Omega\sin\Omega t + b\Omega\cos\Omega t$; and: $x'' = -a\Omega^2\cos\Omega t - b\Omega^2\sin\Omega t$
Substituting and equating coefficients of $\cos\Omega t$ and $\sin\Omega t$ gives the following simultaneous equations: $1 - 2b\Omega + a(\Omega^2 - 10) = 0$ and $b(\Omega^2 - 10) + 2a\Omega = 0$ 'Simplifying' gives:

$$a = -\frac{(\Omega^2 - 10)}{4\Omega^2 + (\Omega^2 - 10)^2} \quad \text{and} \quad b = \frac{2\Omega}{4\Omega^2 + (\Omega^2 - 10)^2}$$

So the general solution becomes: $x = e^{-t}(C\cos 3t + D\sin 3t) + \dfrac{(10 - \Omega^2)\cos\Omega t + 2\Omega\sin\Omega t}{4\Omega^2 + (\Omega^2 - 10)^2}$

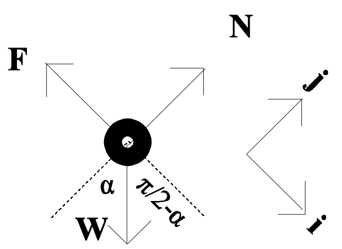
ii Transient part of solution is: $e^{-t}(C\cos 3t + D\sin 3t)$ because as $t \to \infty$ this term $\to 0$. The remaining part is the steady state solution. Now from Handbook page 33 the solution of the form $P\cos\omega t + Q\sin\omega t$ can also be written as $R\cos(\omega t + \phi)$ where $R = \sqrt{P^2 + Q^2}$ and where R is the amplitude. So the amplitude of the steady state solution is:

$$\sqrt{a^2 + b^2} = \sqrt{\frac{(-(\Omega^2 - 10))^2}{(4\Omega^2 + (\Omega^2 - 10)^2)^2} + \frac{4\Omega^2}{(4\Omega^2 + (\Omega^2 - 10)^2)^2}} = \sqrt{\frac{(\Omega^2 - 10)^2 + 4\Omega^2}{(4\Omega^2 + (\Omega^2 - 10)^2)^2}}$$

$$\text{So : Amplitude} = \frac{1}{\sqrt{4\Omega^2 + (\Omega^2 - 10)^2}} = \frac{1}{\sqrt{(10 - \Omega^2)^2 + 4\Omega^2}}$$ as required.

iii $f'(\Omega) = 4\Omega(\Omega^2 - 8)$. Minimum at $f'(\Omega) = 0$. This has solutions: $\Omega = 0$ or $-2\sqrt{2}$ or $+2\sqrt{2}$. The only positive on is $2\sqrt{2}$. From the given original equation we can deduce that $m = 1$; $r = 2$; $k = 10$ and from the general solution we know that $\omega = 3$. Using $\beta = \sqrt{1 - 2\alpha^2}$ we can deduce that for resonance to occur the $\Omega = 6/\sqrt{5} \approx 2.68$. The actual value of Ω we are dealing with $= 2\sqrt{2}$ which ≈ 2.83 (which is close to 2.68) so near this of value $2\sqrt{2}$ we would get resonance.

Q16 i



ii
$\ddot{\mathbf{x}} = \ddot{x}\mathbf{i}$
$\mathbf{N} = |\mathbf{N}|\mathbf{j}$
$\mathbf{F} = -|\mathbf{F}|\mathbf{i}$
$|\mathbf{F}| = \mu|\mathbf{N}|$
$\mathbf{W} = |\mathbf{W}|\sin\alpha\mathbf{i} - |\mathbf{W}|\cos\alpha\mathbf{j} = mg\sin\alpha\mathbf{i} - mg\cos\alpha\mathbf{j}$
Since there is motion we have: $\mathbf{N} + \mathbf{F_1} + \mathbf{F_2} + \mathbf{W} = m\ddot{\mathbf{x}}$
Resolving in **j**-direction: $|\mathbf{N}| = mg\cos\alpha$. So $|\mathbf{F}| = \mu mg\cos\alpha$

Since we have motion then by Newton's Second Law, resolving in **i**-direction gives:
$m\ddot{x}\mathbf{i} = mg\sin\alpha\mathbf{i} - |\mathbf{F}|\mathbf{i} = mg\sin\alpha\mathbf{i} - \mu mg\cos\alpha$. So: $\ddot{x} = a = g\sin\alpha - \mu g\cos\alpha$ (as required)

iii $\ddot{x} = v\dfrac{dv}{dx} = g\sin\alpha - \mu g\cos\alpha \Rightarrow \int v\,dv = g\int(\sin\alpha - \mu\cos\alpha)dx \Rightarrow \dfrac{v^2}{2} = xg(\sin\alpha - \mu\cos\alpha) + C$

Initially when x = 0, v = 0 and so C = 0. So:

$$\frac{v^2}{2} = gx(\sin\alpha - \mu\cos\alpha) \Rightarrow v^2 = 2gx(\sin\alpha - \mu\cos\alpha) \Rightarrow v = \sqrt{2gx(\sin\alpha - \mu\cos\alpha)}$$

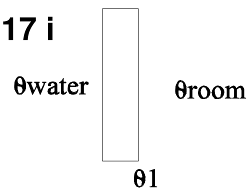
iv $E = KE + U(x)$. $U(x)$ is the potential energy due to gravity from datum at O and so is $-mgx\sin\alpha$. It is negative because m is moving below the datum and is $x\sin\alpha$ because grav. PE is a function of the vertical height above or below the datum. $KE = \frac{1}{2}mv^2$.
So $E = \frac{1}{2}mv^2 - mgx\sin\alpha$ as required.

Using the expression for v found in part iii above we get:

$$v = \sqrt{2gx(\sin\alpha - \mu\cos\alpha)} \Rightarrow E = \tfrac{m}{2}2gx(\sin\alpha - \mu\cos\alpha) - mgx\sin\alpha = -mgx\mu\cos\alpha$$

This is a linear equation with respect to x, the position, with negative gradient and so as x increases, E will decrease, so E is a decreasing function of position.

Q17 i



a $q = \dfrac{kA(\theta_{water} - \theta_1)}{b}$ so $\theta_{water} - \theta_1 = \dfrac{q}{A}\dfrac{b}{k}$

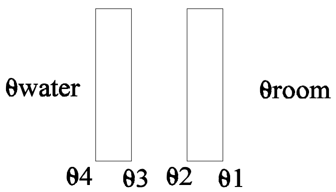
b $q = h_{room}A(\theta_1 - \theta_{room})$ so $\theta_1 - \theta_{room} = \dfrac{q}{A}\dfrac{1}{h_{room}}$

ii Adding the two right hand equations gives:

$$\theta_{water} - \theta_{room} = \frac{q}{A}\left(\frac{1}{h_{room}} + \frac{b}{k}\right) \text{ So: } q = AU(\theta_{water} - \theta_{room}) \text{ where: } U = \left(\frac{1}{h_{room}} + \frac{b}{k}\right)^{-1}$$

This is for one wall and so with all 4 walls the equation becomes: $q = 4AU(\theta_{water} - \theta_{room})$

iii



a $q = \dfrac{kA(\theta_{water} - \theta_3)}{b}$ so $\theta_{water} - \theta_3 = \dfrac{q}{A}\dfrac{b}{k}$

b $q = h_cA(\theta_3 - \theta_2)$ so $\theta_3 - \theta_2 = \dfrac{q}{A}\dfrac{1}{h_c}$

c $q = \dfrac{kA(\theta_2 - \theta_1)}{b}$ so $\theta_2 - \theta_1 = \dfrac{q}{A}\dfrac{b}{k}$

d $q = h_{room}A(\theta_1 - \theta_{room})$ so $\theta_1 - \theta_{room} = \dfrac{q}{A}\dfrac{1}{h_{room}}$

iv Adding the right hand equations gives:

$$\theta_{water} - \theta_{room} = \frac{q}{A}\left(\frac{1}{h_{room}} + \frac{2b}{k} + \frac{1}{h_c}\right) \text{ So: } q = AU(\theta_{water} - \theta_{room}) \text{ where: } U = \left(\frac{1}{h_{room}} + \frac{2b}{k} + \frac{1}{h_c}\right)^{-1}$$

$$\text{So for 4 walls : } q = 4AU(\theta_{water} - \theta_{room})$$

v θ_{water} = 25°C; θ_{room}=19°C; A = 0.4m²; b = 0.007m; k = 0.02Wm^{-1}K^{-1}; h_c = 1.75Wm^{-2}K^{-1}
h_{room} = 150Wm^{-2}K^{-1}; r = 1.5 x 10-8 £J^{-1} (cost of electricity); t = 1 year = 3 x 10^7 s
Cost of running single glazed tank is the cost of q x t J over the year. Substituting gives:

$$q = 4A\left(\frac{1}{h_{room}} + \frac{b}{k}\right)^{-1}(\theta_{water} - \theta_{room}) = 4\times0.4\left(\frac{1}{150} + \frac{0.007}{0.02}\right)^{-1}\times6 \approx 26.92\text{W}$$

$$\text{Energy in year} = E = q\times t = 26.92\times3\times10^7 \approx 807.48\text{MJ}$$

$$\text{So cost} = C = E\times r = 807.48\times10^6\times1.5\times10^{-8} \approx £12.11$$

So the cost of heating the tank for a year with the single walled tank is £12.11 per year.

vi The difference in cost between running the single and double glazed is 12.11 - 3.40 = £8.71. The cost of conversion is £100 so the number of years to recover the cost is 100/8.71 = 11.48 years. (Just under 11 years 6 months.)

Q18 $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}-4 & 6\\ -9 & 11\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}-3\\ 3\end{bmatrix}e^{-t}$ So: $\begin{vmatrix}-4-\lambda & 6\\ -9 & 11-\lambda\end{vmatrix} = 0$

Solving gives λ = 2 and 5 with corresponding eigenvectors $[1\ 1]^T$ and $[2\ 3]^T$ So first part of solution is:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + \beta\begin{bmatrix} 2 \\ 3 \end{bmatrix}e^{5t}$$

Now let:

$$\begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}e^{-t} \Rightarrow -pe^{-t} = -4pe^{-t} + 6e^{-6} - 3e^{-t} \Rightarrow -qe^{-t} = -9pe^{-t} + qe^{-t} + 3e^{-t}$$

$$\text{So: } p = 3 \text{ and } q = 2$$

So general solution is:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + \beta\begin{bmatrix} 2 \\ 3 \end{bmatrix}e^{5t} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}e^{-t}$$

Initially t = 0, x = 0, y = 0 So: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\alpha + 2\beta = -3$ and $\alpha + 3\beta = -2$ Gives: $\alpha = -5$ and $\beta = 1$

So particular solution is: $\begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} 5 \\ 5 \end{bmatrix}e^{2t} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}e^{5t} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}e^{-t}$

Q 19 i If solution is of the form: u(x,t) = X(x)T(t) then:

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t) \quad \text{and} \quad \frac{\partial u}{\partial t} = X(x)T'(t)$$

$$X''(x)T(t) = X(x)T'(t) \Rightarrow \frac{X''(x)T(t)}{X(x)T(t)} = \frac{X(x)T'(t)}{X(x)T(t)} \Rightarrow \frac{X''}{X} = \frac{T'}{T}$$

The only way this can be satisfied for all x and t is if both sides equal the same constant, let us call it μ. So

$$\frac{X''}{X} = \frac{T'}{T} = \mu \quad \text{so } X'' = \mu X \Rightarrow X'' - \mu X = 0$$

Since u(x,t) = X(x)T(t) then u(0,t) = X(0)T(t) = 0 and the only way this can be true for all t is if X(0) = 0. Also u(1,t) = X(1)T(t) = 0 and once again the only way this can be true for all t is if X(1) = 0.

ii If $\mu = 0$ then the solution is X(x) = cx + d. (integrate twice). Using the boundary condition X(0) = 0 = d and so d = 0. Also X(1) = c = 0 and so c = 0 and so the only solution is X(x) = 0

iii If $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ then X(0) = 0 = A + B and so A = -B. Also $X(1) = Ae^{\lambda} - Ae^{-\lambda} = 0$ and the only way this can be so is if A = 0 and so the solution is X(x) = 0.

iv So $\mu < 0$ and so the solution must be of the form $X(x) = A\cos(\lambda x) + B\sin(\lambda x)$. Using the boundary condition: X(0) = Acos(0) + Bsin(0) = 0 and so A = 0. $X(1) = B\sin(\lambda) = 0$. The only way this can be true, excluding the trivial solution of B = 0, is of $\sin\lambda = 0$ and this is only true when $\lambda = r\pi$ with r = 1, 2, ... (if can also be -1, -2, but this is absorbed into B).

Hence: $X(x) = B\sin(r\pi x)$ such that r = 1, 2, ... and so $\mu = r^2 = -1, -4, -9, \ldots$ since $\mu < 0$

v From part i we have that: $T' = \mu T$, but $\mu < 0$ and so $T' + \lambda^2 T = 0$ and this has the solution: $T(t) = Ce^{-\lambda^2 t} = Ce^{-r\pi t}$ with r = 1, 2, ...

Since the solution of the partial differential equation is $u(x,t) = X(x)T(t) = De^{-r\pi t}\sin(r\pi x)$ then as t increases the solution will oscillate with decreasing amplitude towards zero.

vi The family of solutions will be:

$$u(x,y) = \sum_{r=1}^{\infty} D_r \sin(r\pi x)e^{-r\pi t}$$

$$\text{so } u(x,0) = \sum_{r=1}^{\infty} D_r \sin(r\pi x)e^{0} = \sum_{r=1}^{\infty} D_r \sin(r\pi)$$

Q 20 i

$$F_1 = 2y(x+1) \quad \frac{\partial F_1}{\partial y} = 2(x+1) \quad \frac{\partial F_1}{\partial z} = 2z$$

$$F_2 = x(x+\alpha) \quad \frac{\partial F_2}{\partial x} = 2x+\alpha \quad \frac{\partial F_2}{\partial z} = 0$$

$$F_3 = 2xz \quad \frac{\partial F_3}{\partial x} = 2z+\alpha \quad \frac{\partial F_3}{\partial y} = 0$$

$$\textbf{\textit{curl F}} = (\alpha - 2)\boldsymbol{k}$$

ii If F is conservative then curl F = 0 and so $\alpha = 2$
iii Since field is conservative it does not matter which path you take from (0,0,0) to (a,b,c) and so if we let x = at, y = bt and z = ct then as t goes form 0 to 1 we move from (0,0,0) to (a,b,c). Hence $\mathbf{r}(t) = at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k}$ and so $\dot{\mathbf{r}}(t) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$
$\mathbf{F}(\mathbf{r}) = t(2abt + 2b + c^2t)\mathbf{i} + at(at + 2)\mathbf{j} + 2act^2\mathbf{k}$ $\quad \int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^1 \boldsymbol{F}(\boldsymbol{r})\cdot\dot{\boldsymbol{r}}(t)dt = 2ab + a(ab + c^2)$

v Argument as per paper 2002 Q 20 part iv. If you do the maths correctly you do indeed show that **F** = -**grad** U