

## MST207 Past Paper 2002

### Part 1

**Q1** Use integrating factor method:

$$g(x) = 2x \Rightarrow p = e^{\int 2x dx} = e^{x^2}$$

$$\int \left( e^{x^2} \frac{dy}{dx} + e^{x^2} 2xy \right) = \int e^{x^2} e^{-x^2} dx = \int dx \Rightarrow e^{x^2} y = x + C \Rightarrow y = e^{-x^2} (x + C)$$

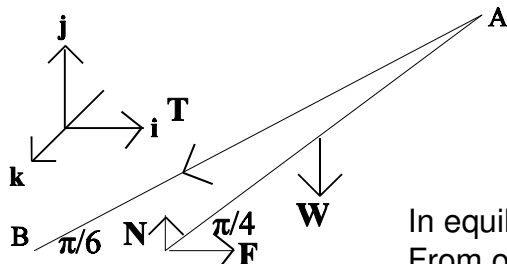
When  $x = 0, y = 1$  then  $C = 1$  So :  $y = e^{-x^2} (x + 1)$

**Q2 i**  $|a| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$        $|b| = \sqrt{(-6)^2 + (-8)^2} = 10$

ii  $\cos \theta = \frac{a \cdot b}{|a||b|} = \frac{1}{30} (2i - j + 2k) \cdot (-6i + 0j - 8k) = -\frac{28}{30} = -\frac{14}{15}$

iii  $c = a \times b = \begin{vmatrix} i & j & k \\ 2 & -1 & 2 \\ -6 & 0 & -8 \end{vmatrix} = (8i + 4j - 6k)$  so  $\hat{c} = \frac{1}{\sqrt{29}} (4i + 2j - 3k)$

### Q3



$$\begin{aligned} \mathbf{W} &= -|W|\mathbf{j} \\ \mathbf{T} &= -|T|\cos(\pi/6)\mathbf{i} - |T|\cos(\pi/3)\mathbf{j} \\ \mathbf{N} &= |N|\mathbf{j} \\ \mathbf{F} &= |F|\mathbf{i} \end{aligned}$$

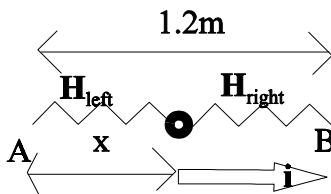
In equilibrium  $\mathbf{W} + \mathbf{T} + \mathbf{N} + \mathbf{F} = 0$  and  $\Gamma_{\mathbf{W}} + \Gamma_{\mathbf{T}} + \Gamma_{\mathbf{N}} + \Gamma_{\mathbf{F}} = 0$   
From origin at O  $\Gamma_{\mathbf{N}} = \Gamma_{\mathbf{F}} = 0$

$\mathbf{r}_{\mathbf{W}} = \frac{1}{2}l \cos(\pi/4)\mathbf{i} + \frac{1}{2}l \sin(\pi/4)\mathbf{j}$  and  $\mathbf{r}_{\mathbf{T}} = l \cos(\pi/4)\mathbf{i} + l \sin(\pi/4)\mathbf{j}$

So:  $\mathbf{r}_{\mathbf{W}} \times \mathbf{W} + \mathbf{r}_{\mathbf{T}} \times \mathbf{T} = 0$  which simplifies to  $|W| = |T|(\sqrt{3} - 1)$  so  $|T| = Mg/(\sqrt{3} - 1)$

### Q4

i



$$\begin{aligned} \mathbf{H}_{\text{left}} &= -k_{\text{left}}(x - l_0)\mathbf{s} = -k(x - l_0)\mathbf{i} \quad (\mathbf{s} = \mathbf{i}) = -10(x - 0.3)\mathbf{i} \\ \mathbf{H}_{\text{right}} &= -k_{\text{right}}(1.2 - x - 0.6)\mathbf{s} = -10(0.6 - x)(-\mathbf{i}) \quad (\mathbf{s} = -\mathbf{i}) = 10(0.6 - x)\mathbf{i} \end{aligned}$$

Where  $x$  is the equilibrium position of the mass, from A.

ii Because no motion:  $\mathbf{H}_{\text{left}} + \mathbf{H}_{\text{right}} = 0$  giving:  $-10(x - 0.3)\mathbf{i} + 10(0.6 - x)\mathbf{i} = 0$   
 $x - 0.3 = 0.6 - x$  giving:  $2x = 0.9$  and so the equilibrium position,  $x = 0.45$  m from A.

**Q5** Given:  $x = 69.76$ m;  $h = 2$ m;  $\theta = 45^\circ$ ; Know:  $x = ut \cos \theta$ ;  $y = h + ut \sin \theta - \frac{1}{2}gt^2$

Trajectory is:  $y = h + x \tan \theta - \frac{gx^2}{2u^2} (1 + \tan^2 \theta)$  and when the javelin lands,  $y = 0$ .

i Substituting gives:  $u = 25.79 \text{ms}^{-1}$

ii Using  $x = ut \cos \theta$  and substituting gives  $t = 3.8$ s

**Q6**

$$\begin{array}{l}
 R_1 \left[ \begin{array}{ccc|c} 1 & -2 & -2 & 1 \end{array} \right] \quad R_1 \left[ \begin{array}{ccc|c} 1 & -2 & -2 & 1 \end{array} \right] \quad R_1 \left[ \begin{array}{ccc|c} 1 & -2 & -2 & 1 \end{array} \right] \\
 R_2 \left[ \begin{array}{ccc|c} 2 & 3 & 1 & 4 \end{array} \right] \quad R_{2a}=R_2-2R_1 \left[ \begin{array}{ccc|c} 0 & 7 & 5 & 2 \end{array} \right] \quad R_{2a} \left[ \begin{array}{ccc|c} 0 & 7 & 5 & 2 \end{array} \right] \\
 R_3 \left[ \begin{array}{ccc|c} -3 & 20 & 18 & 3 \end{array} \right] \quad R_{3a}=R_3+3R_1 \left[ \begin{array}{ccc|c} 0 & 14 & 12 & 6 \end{array} \right] \quad R_{3b}=R_{3a}-2R_{2a} \left[ \begin{array}{ccc|c} 0 & 0 & 2 & 2 \end{array} \right]
 \end{array}
 \quad \begin{array}{l}
 x_3 = 1 \\
 x_2 = -3/7 \\
 x_1 = 15/7
 \end{array}$$

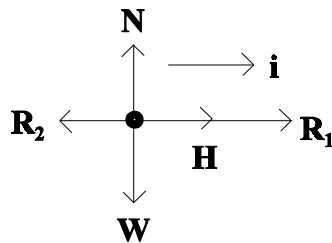
**Q7 i** 4/3    **ii**  $[1 \ 2]^T$  Question is identical to Q7 of 2001 paper.

**Q8** Potential energy datum is the natural length of spring,  $l_0$

Initially  $U_{\text{init}} = U_{\text{grav}} + U_{\text{spring}} = -mg(l_{\text{eq}} - l_0) + \frac{1}{2}k(l_{\text{eq}} - l_0)^2$  where  $l_{\text{eq}}$  is the equilibrium position of the spring and so  $(l_{\text{eq}} - l_0)$  is the extension from hanging the weight.  $U_{\text{grav}}$  is -ve because below datum. When the mass is pulled down from the equilibrium point by an amount  $x$ , the total potential energy is now:  $U_{\text{final}} = -mg(l_{\text{eq}} - l_0 + x) + \frac{1}{2}k(l_{\text{eq}} - l_0 + x)^2$   
 So change,  $\Delta U$  in PE is  $U_{\text{final}} - U_{\text{init}} = -mg(l_{\text{eq}} - l_0 + x) + \frac{1}{2}k(l_{\text{eq}} - l_0 + x)^2 - (-mg(l_{\text{eq}} - l_0) + \frac{1}{2}k(l_{\text{eq}} - l_0)^2)$   
 This simplifies to:  $\Delta U = kx(l_{\text{eq}} - l_0 + \frac{1}{2}x) - mgx$

**Q9**

**i**



$$\begin{aligned}
 m\ddot{x}i &= H + R_1 + R_2 \quad (\text{resolving in } i\text{-direction}) \\
 H &= -k(x - l_0)i \\
 R_1 &= -r_1\dot{x}i \\
 R_2 &= -r_2\dot{x}i \\
 m\ddot{x}i &= -k(x - l_0)i - r_1\dot{x}i - r_2\dot{x}i \\
 \text{Hence: } m\ddot{x} + (r_1 + r_2)\dot{x} + kx &= -kl_0
 \end{aligned}$$

**ii** To be damped then  $r^2 < 4mk$  so  $(r_1 + r_2)^2 / (4k) < m$

**Q10 i** By the principle of conservation of Momentum:  $m\mathbf{u} + 2m\mathbf{v} = 3m\mathbf{w}$

(I am grateful to Nicola Braithwaite and Sharon Horne for corrections to this question)  
 $m\mathbf{u}/\sqrt{2}(\mathbf{i} + \mathbf{j}) + 2m\mathbf{v}\sqrt{2}(\mathbf{i} - \mathbf{j}) = 3m(\mathbf{w}_1\mathbf{i} + \mathbf{w}_2\mathbf{j})$ . Resolving in  $i$ - and  $j$ -direction separately gives:

$$w_1 = \frac{(u+2v)}{3\sqrt{2}} \quad w_2 = \frac{(u-2v)}{3\sqrt{2}} \quad \text{so } \mathbf{w} = \frac{1}{3\sqrt{2}}((u+2v)\mathbf{i} + (u-2v)\mathbf{j})$$

**ii** K.E. before =  $\frac{1}{2}m\mathbf{u}^2 + \frac{1}{2} \cdot 2m\mathbf{v}^2$  where  $|\mathbf{u}|^2 = u^2$  and  $|\mathbf{v}|^2 = v^2$

K.E. after =  $\frac{1}{2} \cdot 3m\mathbf{w}^2$  where  $|\mathbf{w}|^2 = (u^2 + 4v^2)/9$ . K.E. after =  $\frac{1}{2} \cdot 3m(u^2 + 4v^2)/9$ . So loss of K.E. is  $\frac{1}{2}m\mathbf{u}^2 + \frac{1}{2} \cdot 2m\mathbf{v}^2 - \frac{1}{2} \cdot 3m(u^2 + 4v^2)/9 = \frac{2}{3}(u^2 + v^2)$  Joules.

**Q11 i** Lin. momentum,  $\mathbf{p} = m\dot{\mathbf{r}} = m\dot{x}\mathbf{i} + 2m\dot{y}\mathbf{j} + m\dot{z}\mathbf{k}$ . Angular momentum,  $\ell = \mathbf{r} \times \mathbf{p} = -m\dot{z}y\mathbf{i} + m\dot{z}x\mathbf{j} + (2m\dot{y}x - m\dot{x}y)\mathbf{k}$

**ii** External force,  $\mathbf{F} = \dot{\mathbf{p}} = 2m\dot{y}\mathbf{j}$

**Q12**  $f(x,y) = x^3 + x^2y - xy^2 - y^3 - y^2$ ;  $f_x = 3x^2 + 2yx - y^2$ ;  $f_y = x^2 - 2xy - 3y^2 - 2y$

Substituting  $(-3/16, -9/16)$  into each of the above gives  $f_x = f_y = 0$  and so this point is a stationary point.  $A = f_{xx} = 6x + 2y$ ;  $C = f_{yy} = -2x - 6y - 2$ ;  $B = f_{xy} = 2x - 2y$   
 $AC - B^2$  at  $(-3/16, -9/16) = -51 < 0$  hence stationary point is a Saddle Point.

**Q13**  $Y_{40} - Y_{20} \approx C(h_{40}^2 - h_{20}^2)$  (since 2nd order method)  $C \approx -0.061$ .

For 5 d.p. accuracy  $h^2 < (0.5 \times 10^{-5}) / |C|$  so max step size  $h < 0.0091$

**Q14**

$$A = \int_{-1}^1 \int_{y=0}^{y=1-x^2} (y^2) dy dx = \frac{1}{3} \int_{-1}^1 (1 - x^2)^3 dx = \frac{2}{3} \int_0^1 (1 - x^2)^3 dx = \frac{32}{105}$$

## Part 2

**Q15 i** Characteristic equation is  $\lambda^2 + 2\lambda + 10$  which has solutions  $-1 \pm 3i$  and so solution to homogeneous equation is:  $y = e^{-t}(C \cos 3t + D \sin 3t)$

Try:  $yp = ke^{-t}$

$$y' = -ke^{-t}$$

$$y'' = ke^{-t}$$

So  $ke^{-t} - 2ke^{-t} + 10ke^{-t} = 18ke^{-t}$  which gives  $k = 2$ . So now:  $y = e^{-t}(C \cos 3t + D \sin 3t) + 2e^{-t}$

Try:  $yp = a \cos 2t + b \sin 2t$

$$y' = -2a \sin 2t + 2b \cos 2t$$

$$y'' = -4a \cos 2t - 4b \sin 2t$$

So:  $\cos 2t(6a + 4b) + \sin 2t(-4a + 6b) = 14 \cos 2t - 18 \sin 2t$  which gives  $a = 3$ ;  $b = -1$

So now  $y = e^{-t}(C \cos 3t + D \sin 3t) + 2e^{-t} + 3 \cos 2t - \sin 2t$

When  $t = 0$ ;  $y = -3$ , substituting gives  $-3 = C + 2 + 3$  so  $C = -8$

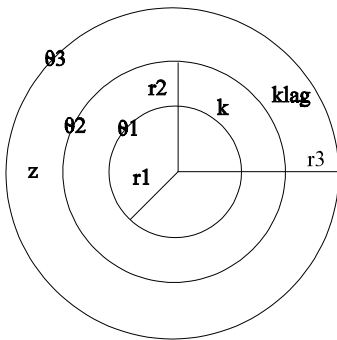
So now  $y = e^{-t}(-8 \cos 3t + D \sin 3t) + 2e^{-t} + 3 \cos 2t - \sin 2t$

When  $t = 0$ ;  $y' = 16$  which after much algebra gives  $D = 4$ .

So particular solution is:  $y = e^{-t}(-8 \cos 3t + 4 \sin 3t) + 2e^{-t} + 3 \cos 2t - \sin 2t$

**ii** Long term  $t \rightarrow \infty$  and so  $y \rightarrow 3 \cos 2t - \sin 2t$  which is a sinusoidal curve with period  $\pi$  (from  $2\pi/2$ ) and amplitude  $\sqrt{10}$  (from  $\sqrt{3^2 + 1^2}$ ).

## Q16 i



By convection water to pipe:

$$q = 2\pi h_{in} r_1 l (\theta_{in} - \theta_1) \quad \text{so: } \theta_{in} - \theta_1 = \frac{q}{2\pi l} \times \frac{1}{h_{in} r_1}$$

By conduction through pipe:

$$q = \frac{2\pi k l (\theta_1 - \theta_2)}{\ln\left(\frac{r_2}{r_1}\right)} \quad \text{so: } \theta_1 - \theta_2 = \frac{q}{2\pi l k} \ln\left(\frac{r_2}{r_1}\right)$$

By conduction through lagging:

$$q = \frac{2\pi k_{lag} l (\theta_2 - \theta_3)}{\ln\left(\frac{r_2+z}{r_1}\right)} \quad \text{so: } \theta_2 - \theta_3 = \frac{q}{2\pi l k_{lag}} \ln\left(\frac{r_2+z}{r_1}\right)$$

By convection lagging to air:  $q = 2\pi h_{out} (r_2+z) l (\theta_3 - \theta_{out})$  so:  $\theta_3 - \theta_{out} = \frac{q}{2\pi l h_{out} (r_2+z)}$

Adding together all the right hand side equations gives:

$$\theta_{in} - \theta_{out} = \frac{q}{2\pi l} \left( \frac{1}{h_{in} r_1} + \frac{1}{k} \ln\left(\frac{r_2}{r_1}\right) + \frac{1}{k_{lag}} \ln\left(\frac{r_2+z}{r_1}\right) + \frac{1}{h_{out} (r_2+z)} \right)$$

$$\text{So: } q = 2\pi l \left( \frac{1}{h_{in} r_1} + \frac{1}{k} \ln\left(\frac{r_2}{r_1}\right) + \frac{1}{k_{lag}} \ln\left(\frac{r_2+z}{r_1}\right) + \frac{1}{h_{out} (r_2+z)} \right)^{-1} (\theta_{in} - \theta_{out})$$

**ii** The above equation is equivalent to  $q = 2\pi l U (\theta_{in} - \theta_{out})$ . So  $q$  is a maximum when  $U$  is a maximum or when  $1/U$  is a minimum. Let  $R = 1/U$  then:

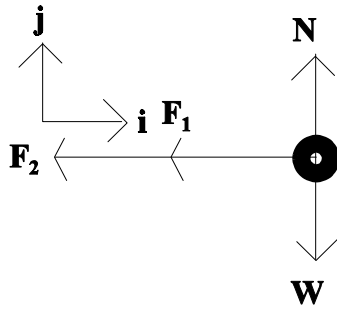
$$R = \left( \frac{1}{h_{in} r_1} + \frac{1}{k} \ln\left(\frac{r_2}{r_1}\right) + \frac{1}{k_{lag}} \ln\left(\frac{r_2+z}{r_1}\right) + \frac{1}{h_{out} (r_2+z)} \right) \Rightarrow \frac{dR}{dz} = 0 + 0 + \frac{1}{k_{lag}} \frac{1}{(r_2+z)} - \frac{1}{h_{out} (r_2+z)^2}$$

$$\text{If } \max \frac{dR}{dz} = 0 = \frac{1}{k_{lag} (r_2+z)} - \frac{1}{h_{out} (r_2+z)^2} \Rightarrow \frac{1}{k_{lag} (r_2+z)} = \frac{1}{h_{out} (r_2+z)^2} \Rightarrow (r_2+z) = \frac{k_{lag}}{h_{out}} \Rightarrow z = \frac{k_{lag}}{h_{out}} - r_2$$

and since  $z$  is the thickness of the lagging, this is the value at which the transfer of heat,  $q$ , is a maximum.

**Q17**

i



$$\begin{aligned} \ddot{\mathbf{x}} &= \ddot{x}\mathbf{i} \\ \mathbf{N} &= |\mathbf{N}|\mathbf{j} \\ \mathbf{F}_1 &= -|\mathbf{F}_1|\mathbf{i} = -k_1v\mathbf{i} \\ \mathbf{F}_2 &= -|\mathbf{F}_2|\mathbf{i} = -k_2\mathbf{i} \\ \mathbf{W} &= -|\mathbf{W}|\mathbf{j} = -mg\mathbf{j} \end{aligned}$$

Since there is motion we have:  $\mathbf{N} + \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{W} = m\ddot{\mathbf{x}}$   
Resolving in i-direction gives:  $-k_1v\mathbf{i} - k_2\mathbf{i} = m\ddot{x}\mathbf{i}$

So:

$$m\ddot{x} = -k_1v - k_2 \Rightarrow m\dot{x} = m\frac{dv}{dt} = -(k_1v + k_2) \Rightarrow -m\int\frac{dv}{k_1v + k_2} = \int dt \Rightarrow -\frac{m}{k_1}\ln(k_1v + k_2) = t + C$$

Initially when  $t = 0, v = u_0$

$$C = -\frac{m}{k_1}\ln(k_1u_0 + k_2) \Rightarrow t = -\frac{m}{k_1}\ln(k_1v + k_2) + \frac{m}{k_1}\ln(k_1u_0 + k_2) = \frac{m}{k_1}\ln\left(\frac{k_1u_0 + k_2}{k_1v + k_2}\right)$$

When the plane comes to a rest  $v = 0$  and  $t = T$  so:  $T = \frac{m}{k_1}\ln\left(\frac{k_1u_0 + k_2}{k_2}\right) = \frac{m}{k_1}\ln\left(1 + \frac{k_1}{k_2}u_0\right)$

ii  $m\dot{x} = -k_1v - k_2 \Rightarrow mv\frac{dv}{dx} = -(k_1v + k_2) \Rightarrow \int\frac{mvdv}{k_1v + k_2} = -\int dx$

I am grateful to Peter Baxendall for the following solution which is so much simpler and more elegant than my original attempt.

$$\frac{mv}{k_1v + k_2} = \frac{m}{k_1}\left(\frac{k_1v + k_2 - k_2}{k_1v + k_2}\right) = \frac{m}{k_1}\left(1 - \frac{k_2}{k_1v + k_2}\right) \Rightarrow \int_{u_0}^0\frac{mvdv}{k_1v + k_2} = \int_{u_0}^0 m\left(1 - \frac{k_2}{k_1v + k_2}\right)dv = -\int_0^X dx$$

$$-X = \frac{m}{k_1}\left(-u_0 - \frac{k_2}{k_1}\ln\left(\frac{k_2}{k_1u_0 + k_2}\right)\right) \Rightarrow \text{So: } X = \frac{m}{k_1}\left(u_0 + \frac{k_2}{k_1}\ln\left(\frac{k_2}{k_1u_0 + k_2}\right)\right)$$

Where X is the distance it has travelled when it comes to rest at  $v = 0$

**Q18 i**  $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2x - 2y^2 \\ 8y - x^2 \end{bmatrix}$

ii Equilibrium points found from solving:  $2x - 2y^2 = 0$  and  $8y - x^2 = 0$ . One solution is obviously  $(0,0)$ . From first equation  $x = y^2$ , substituting in second gives  $y = 0$  or  $y = 2$  (there are no other real solutions). Putting  $y = 2$  back into first, gives  $x = 4$ . So the other point is  $(4,2)$ .

iii  $J(x,y) = \begin{bmatrix} 2 & -4y \\ -2x & 8 \end{bmatrix}$  so  $\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 2 & -4y \\ -2x & 8 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

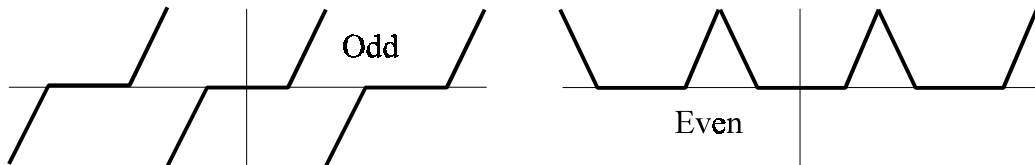
Point  $(0,0)$  gives:  $J(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$  so eigenvalues are 2 and 8 with eigen vectors  $[1,0]$  and  $[0,1]$ . Since real and same sign is an Unstable Source.

Point  $(4,2)$  gives:  $J(4,2) = \begin{bmatrix} 2 & -8 \\ -8 & 8 \end{bmatrix}$  so eigenvalues are  $5 + \sqrt{73}$  and  $5 - \sqrt{73}$ . Corresponding eigenvectors are:  $[3 - \sqrt{73}, 8]$  and  $[3 + \sqrt{73}, 8]$ .

These are real and opposite sign and so is Unstable Saddle. See Handbook/Unit for sketches of such points.

**Q19 i**

$$f_{\text{odd}}(t) = \begin{cases} 2t - 1 & 1/2 < t \leq 1 \\ 0 & -1/2 < t \leq 1/2 \\ 2t + 1 & -1 \leq t \leq -1/2 \end{cases} \quad f_{\text{even}}(t) = \begin{cases} 2t - 1 & 1/2 < t \leq 1 \\ 0 & -1/2 < t \leq 1/2 \\ -(2t + 1) & -1 \leq t \leq -1/2 \end{cases}$$



ii 
$$A_0 = \int_0^1 f(t) dt = \int_0^{1/2} 0 dt + \int_{1/2}^1 (2t-1) dt = t^2 - t \Big|_{1/2}^1 = 1/4$$

iii 
$$A_r = 2 \int_0^1 f(t) \cos(r\pi t) dt = 2 \int_{1/2}^1 (2t-1) \cos(r\pi t) dt = \frac{4}{r^2 \pi^2} \left( (-1)^r - \cos\left(\frac{r\pi}{2}\right) \right)$$

(Using integration by parts or the hint given in the question)

iv

$$B_r = 2 \int_0^1 f(t) \sin(r\pi t) dt = 2 \int_{1/2}^1 (2t-1) \sin(r\pi t) dt = \frac{2(-1)^{r+1}}{r\pi} - \frac{4}{r^2 \pi^2} \sin\left(\frac{r\pi}{2}\right)$$

(Using integration by parts or the hint given in the question)

**Q20 i**  $F_1 = 2(x-yz) \quad F_{1x} = -2y \quad F_{1y} = -2z \quad \text{So } \mathbf{curl} \mathbf{F}_1 = \mathbf{0}$

$F_2 = 2(y-xz) \quad F_{2x} = -2z \quad F_{2z} = -2x$

$F_3 = -2xy \quad F_{3x} = -2y \quad F_{3y} = -2x$

For  $\mathbf{F}_2$  only difference is  $F_{3x} = 2y \quad F_{3y} = 2x \quad \text{So } \mathbf{curl} \mathbf{F}_2 = 4xi - 4yj$

ii  $\mathbf{F}_1$  is conservative because  $\mathbf{curl} \mathbf{F}_1 = \mathbf{0}$  and for same reason  $\mathbf{F}_2$  is not conservative.

iii  $x = at; y = bt; z = ct$  So  $\mathbf{F}_1(t) = 2(at - bct^2)\mathbf{i} + 2(bt - act^2)\mathbf{j} - 2abt^2\mathbf{k}$  and  $\dot{\mathbf{r}} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

$$\int_C \mathbf{F}_1 \cdot d\mathbf{r} = \int_0^1 ((2(x-yz)\mathbf{i} + 2(y-xz)\mathbf{j} - 2xyz\mathbf{k}) \cdot (a\mathbf{i} + b\mathbf{j} + c\mathbf{k})) dt = a^2 + b^2 - 2abc$$

iv This is the negative potential field  $-U = -(U(a,b,c) - U(0,0,0))$ . If we let datum be  $(0,0,0)$  then this simplifies to  $-U(a,b,c)$ . So  $-U(x,y,z) = x^2 + y^2 - 2xyz$ . So  $U(x,y,z) = -(x^2 + y^2 - 2xyz)$ . Now  $\mathbf{grad} U = -(2(x - yz)\mathbf{i} + 2(y - xz)\mathbf{j} - 2xyz\mathbf{k})$ . So  $-\mathbf{grad} U = \mathbf{F}_1$