MS221 – 2008 Solutions

Qn.1 (a) $u_2 = 25.5$

(b) Solve $2r^2 + r - 6=0$, i.e. $(2r-3)(r+2)=0$ to give r = 1.5 and r = -2 as roots.

General solution is

$$u_n = A(1.5)^n + B(-2)^n$$

Use u_0 and u_1: A + B = 9
3A – 4B = 6

Solution is A = 6, B = 3, so closed form is

$$u_n = 6(1.5)^n + 3(-2)^n \quad (n=0,1,2,..)$$

Qn.2 (a) eqn. is $\frac{x^2}{9} + \frac{y^2}{5} = 1$ which is an ellipse in standard position with a=3,b=$\sqrt{5}$.

(b) (i) $e^2 = 1-5/9 = 4/9$, so e = 2/3.
(ii) Foci $(\pm 2, 0)$
(iii) Directrices $x = \pm 4.5$

Qn.3 (a)(i) $(x,y) \mapsto (x+2, y-4)$ and

$(x,y) \mapsto (x\cos(2\pi/3) + y\sin(2\pi/3), x\sin(2\pi/3) - y\cos(2\pi/3))$

that is $(x,y) \mapsto (-\frac{1}{2}x + \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x + \frac{1}{2}y)$

(ii) $(x,y) \mapsto (-\frac{x+2}{2} + \frac{\sqrt{3}(y-4)}{2}, \frac{\sqrt{3}(x+2)}{2} + \frac{y-4}{2})$

that is $(x,y) \mapsto (\frac{1}{2}(-x + \sqrt{3}y) - (1+2\sqrt{3}),$
$\frac{1}{2}(\sqrt{3}x + y) - (2-\sqrt{3}))$

(b) $\cos(2x)=2\cos^2 x - 1$, so

$\cos^2(\pi/12) = \frac{1}{2}(1+\cos(\pi/6)) = \frac{1}{2}(1+\frac{\sqrt{3}}{2}) = \frac{1}{4}(2+\sqrt{3})$

$\cos(\pi/12) = \frac{1}{2}(\sqrt{2+\sqrt{3}})$

Qn.4 (a) $f'(x) = -\frac{2}{3}x + 2$

$|f'(-1)| = 2\frac{2}{3}$ >1 so repelling ;

$|f'(4)| = |-\frac{2}{3}|$ <1 so attracting.

(b) Sketch a parabola through (0,4/3), (-1,-1), (4,4), $(3 \pm \sqrt{13}, 0)$ with its vertex at (3,13/3), and also show the line y=x.

Qn.5 (a) $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ y-shear with factor 3

invariant line : y=0

(Note : y=0 is the only line for which strict, that is point to point, invariance applies. Less strictly, any line of the form y=k maps onto itself, and is in that sense invariant.)

(b) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ reflection in y=x, invariant line : y=x

(Note : y=-x is also an invariant line in the less strict sense.)

(c) $\begin{pmatrix} 7 \\ 14 \end{pmatrix}$ flattening onto y=2x, invar. line:y=2x

Qn.6 (a) char eqn. is $k^2 - 2k - 15 = 0$
that is $(k+3)(k-5) = 0$

k=-3 : e-line is 10x + 5y = 0, i.e. y = -2x

k=5 : e-line is 2x + 5y = 0, i.e. y = -0.4x

possible e-vectors are $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ resp.

Qn.7 (a) $f'(x) = \frac{e^{3x}(3\sin x - \cos x)}{\sin^2 x}$

(b) $g'(x) = \frac{1}{2x\sqrt{\ln(x)}}$

Qn.8 (a) $\frac{1}{5}(x+1)\sin(5x) + \frac{1}{25}\cos(5x) + c$

(b) substitute $u=e^{x/2}$ so $du = \frac{1}{2}e^{x/2}dx$

integral = $2\arctan(e^{x/2}) + c$

Qn.9 (a) (i) $1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3$

(ii) $1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$

(b) $\frac{1}{2}x^2 - \frac{1}{4}x^3$

(c) h′ (0) = coeff. of x = 0, so h(x) has a stationary point at x = 0

h″ (0) = +1 (2 times coeff. of x^2), so stationary pt at x = 0 is a minimum.

Qn.10 (a) modulus = $\sqrt{2}$ argument = $\frac{\pi}{4}$

(b) $< \sqrt{2}, \frac{\pi}{4} >$.

(c) $z^6 = < 2^3, \frac{6\pi}{4} > = < 8, \frac{3\pi}{2} > = -8i$

(d) $-8i + (1 - i) = 1 - 9i$

Qn.11 (a) 30 = 4.7 + 2
7 = 3.2 + 1
1 = 7 – 3.2
= 7 – 3[30 –4.7] =7 – 3.30 + 12.7
= 13.7 in $\mathbb{Z}_{30}$ since 3.30 = 0

so 7^{-1} = 13 in $\mathbb{Z}_{30}$.

(b) $4^{13}=2$ (in mod 31 arithmetic $4^2=16$, $4^4=256=8$, $4^8=64=2$ so $4^{13}=2.8.4=64=2$).

Qn.12

(a) $b(n) \wedge c(n)$

(b) (i) any integer 3*n* where *n* is odd

(ii) $a(n) \Rightarrow b(n) \vee c(n)$

An even number is divisible by either 3 or 4. Not true for all *n*, e.g. *n*=2.

(Note : Line 1 of the qn says "where $n \varepsilon \boldsymbol{N}$" not "for all $n \varepsilon \boldsymbol{N}$" , so the converse is "for $n \varepsilon \boldsymbol{N}$..." i.e. a function of *n*, and thus another variable proposition (see D4 p.23). So strictly speaking, the answer "False" is incorrect, although it would probably be accepted.)

Qn.13 Use notation of p. 49 in Handbook.

(a) Asymptotes are $y = \pm \frac{2}{3}x$

(b) sketch should show vertices at (3,0) and (-3,0) .

(c) since A=C $\theta = \frac{\pi}{4}$

$A' = \frac{1}{2}(A + B + C) = -8$

$C' = \frac{1}{2}(A - B + C) = 18$

$F' = 72$

so L is $-8x^2 + 18y^2 + 72 = 0$, that is $\frac{x^2}{9} - \frac{y^2}{4} = 1$

showing that $L = r_\theta(K)$

(d) L is a hyperbola with axes of symmetry $x=y$ and $x=-y$, and asymptotes approximately coinciding with the x and y axes.

(e) eccentricity is $\sqrt{1 + \frac{4}{9}} = \frac{1}{3}\sqrt{13}$.

(Note : The exact equations of the asymptotes can be obtained by calculating their gradients as

$\arctan\left(\frac{\frac{2}{3}+1}{1-\frac{2}{3}}\right)$ and $\arctan\left(\frac{-\frac{2}{3}+1}{1+\frac{2}{3}}\right)$

= arctan (5) and arctan (0.2) respectively, so eqns. are $y = 5x$ and $y = 0.2x$)

(f) $5x^2 - 26xy + 5y^2 + 72 = 0$

Rotating by π simply interchanges the two branches of the hyperbola, leaving its overall shape unaltered.

Qn.14 (a) (i) $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ (scaling)

(ii) $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ (x-shear, factor -1)

(iii) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (anti-clockwise rotation)

(iv) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

rotation following shear

(v) $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 3 & -2 \end{pmatrix}$

B -> D following scaling

(vi) $\begin{pmatrix} 0 & -2 \\ 3 & -2 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} -2 & 2 \\ -3 & 0 \end{pmatrix}$

inverse of (v)

(b) (i) $x \mapsto x + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

(ii) $x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\{x + \begin{pmatrix} 2 \\ -1 \end{pmatrix}\} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}x + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Qn.15 (a) (i) $f'(x) = e^{-px}(\sec^2 x - p\tan x)$

then use the identity $\sec^2 x = \tan^2 x + 1$

(ii) The quadratic for tan$x$ in part (i) has two distinct real roots only if $p^2 - 4 > 0$, hence there are two stationary points if and only if p>2 or p<-2.

(iii) If p=2, the single solution is tan$x$=1, so x = π/4, and the coordinates of the stationary point are $(\frac{\pi}{4}, e^{-\pi/2})$

The graph goes through the origin and is asymptotic to both x = -π/2 and x = π/2. It has a point of inflection (i.e. f″ (x)=0 so there is neither a maximum nor a minimum) at x = π/4.

(b) (i) Area = $\int_0^{\sqrt{3}} h(x)dx$

substitute $u=1+x^2$ so $du = 2x\,dx$, limits=1 to 4

area= $\int_1^4 \frac{du}{2\sqrt{u}} = \left[\sqrt{u}\right]_1^4 = 2 - 1 = 1$

(ii) volume = $\pi\int_0^{\sqrt{3}} [h(x)]^2 dx = \pi\int_0^{\sqrt{3}} \frac{x^2}{1+x^2}dx$

$= \pi\int_0^{\sqrt{3}} \{1 - \frac{1}{1+x^2}\}dx = \pi\left[x - \arctan x\right]_0^{\sqrt{3}}$

$= \pi\left[\sqrt{3} - \arctan\sqrt{3}\right] = \pi\left[\sqrt{3} - \frac{\pi}{3}\right]$ = 2.1515

Qn.16 (a) (i) Write

$\{r_0, r_\pi, q_0, q_{\pi/2}\}$={I, H, X,Y}

A's elements are {I,H,X,Y}

B's elements are {I,H}

(ii)

	I	X	Y	H
I	I	X	Y	H
X	X	I	H	Y
Y	Y	H	I	X
H	H	Y	X	I

or equi-val-ently

	I	H	X	Y
I	I	H	X	Y
H	H	I	Y	X
X	X	Y	I	H
Y	Y	X	H	I

.

(iii) Yes. {I,H}are a subset of A's elements which satisfy the group table

	I	H
I	I	H
H	H	I

(b) (i) H is not a group because 2x6 = 0 in modulo 12 arithmetic, and thus the closure group axiom is not fulfilled.

(ii)

·	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

(iii) isomorphism is

(ω(1)=I, ω(5)=X, ω(7)=Y, ω(11)=H)