MS221 – 2007 Solutions

Qn.1 (a) $u_2 = 19$

(b) Solve $3r^2 - 8r - 3 = 0$, i.e. $(3r+1)(r-3)=0$ to give r=3 and r = -1/3 as roots.

General solution is

$$u_n = A(3^n) + B(-\tfrac{1}{3})^n$$

Use u_0 and u_1: A + B = 11

3A –1/3 B = 3

Solution is A =2, B=9, so closed form is

$u_n = 2(3^n) + 9(-\tfrac{1}{3})^n$ (n=0,1,2,..)

Qn.2 (a) eqn. is $\frac{x^2}{25} + \frac{y^2}{9} = 1$, ellipse.

(b) (i) $e^2 = 1 - 9/25 = 16/25$, so e=4/5.

(ii) Foci (± 4,0) (iii) directrices $x = \pm 25/4$.

Qn.3 (a) $\sin\theta = \frac{3}{\sqrt{13}}$ $\cos\theta = \frac{2}{\sqrt{13}}$

(b) $\cos 2\theta = -5/13$ $\sin 2\theta = 12/13$

(c) $(x, y) \mapsto \left(\frac{-5x + 12y}{13}, \frac{12x + 5y}{13} \right)$

Qn.4

(a) ABC, ABD, ABE, ACD, ACE, ADE

(b) 1140 (c) -18677760

Qn.5 (a) $y = \frac{1}{2}x$ (k=3), $y = -\frac{1}{3}x$ (k=1/2)

(b) (iii) is OK.

(i) has plots on alternate sides of the origin although 3 > 0.

(ii) plots are not increasing threefold between x_1 and x_2.

Qn. 6 (a) $f = \begin{pmatrix} 2 & -2 \\ 1 & -3 \end{pmatrix}$ $g = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$

$g \circ f = \begin{pmatrix} 2 & -2 \\ -5 & 3 \end{pmatrix}$

(b) $(f \circ h)(\mathbf{x}) = \begin{pmatrix} 4 & 2 \\ 8 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 6 \\ 7 \end{pmatrix}$

Qn.7 (a) $2x \arccos x - \frac{x^2 - 1}{\sqrt{1 - x^2}}$

$= 2x \arccos x + \sqrt{1 - x^2}$

(b) $3e^{-3x} \operatorname{cosec}^2 (e^{-3x})$

Qn.8 (a) $\frac{x^6}{6}\ln(2x) - \frac{x^6}{36} + c$

(b) $\frac{2(1+x)^{3/2}}{3} - 2(1+x)^{1/2} + c$

Qn.9 (a) $-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$

(b) $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

$-xe^x = -x - x^2 - \frac{x^3}{2} - \frac{x^4}{6}$

$\ln(1-x) = +x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$

so $g(x) = 1 + x + \frac{x^4}{8}$

(c) 1 + 0.2 – 0.0002 = 1.200 to 4 sig. figs.

Qn.10 (a) $z^2 - 4z + 8$

(b) -5 since last factor must be (z+5) to make constant term 40.

(c) $< 2\sqrt{2}, (-\frac{\pi}{4}) >$

Qn.11 (a) (i) yes (3 divides digit sum)

(ii) no because last two digits 14 are not divisible by 4.

(b) 3 (in mod 30 arithmetic 27=-3, 27^2=9, 27^3= -27=3, 27^4=81= -9, 27^8=81= -9, 27^{11}=(-9)(3)= -27=3).

Qn.12 (a) (i) false, example=any odd multiple of 10. (ii) true

(b) $d(n) \Rightarrow (a(n) \wedge b(n))$ example=any odd multiple of 20.

(c) $c(n) \wedge d(n)$

Qn.13 Use p. 49 in Handbook throughout qn.

(a) Asymptotes are $y = \pm \frac{4}{3}x$.

(b) sketch should show vertices at (3,0) and (-3,0) .

(c) $\theta = \frac{1}{2}\arctan\left(\frac{50\sqrt{3}}{39 - (-11)} \right) = \frac{\pi}{6}$

$A' = 39\left(\frac{3}{4}\right) + 50\sqrt{3}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - 11\left(\frac{1}{4}\right)$=256/4

$C' = 39\left(\frac{1}{4}\right) - 50\sqrt{3}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - 11\left(\frac{3}{4}\right)$=-144/4

F' = -576

so L is $64x^2 - 36y^2 - 576 = 0$ which is equivalent to K on division by 576.

(d) sketch should show L identical to K but with its axes rotated anticlockwise by $\pi/6$.

Axes of symmetry are $y = \pm \frac{1}{\sqrt{3}}x$. L’s asymptotes should be close to the x and y axes.

(Note : This can be confirmed by showing that the aymptotes have gradients

$\arctan\left(\frac{4/3 + 1/\sqrt{3}}{1 - \left(4/3\right)\left(1/\sqrt{3}\right)} \right)$ = arctan (8.300) approx, and

$\arctan\left(\frac{-4/3 + 1/\sqrt{3}}{1 + \left(4/3\right)\left(1/\sqrt{3}\right)} \right)$ = arctan (-0.254) approx

so eqns. are approx $y = 8.3x$ and $y = -0.254x$)

(e) $r_{-\vartheta}(K)$ is $39x^2 - 50\sqrt{3}xy - 11y^2 = 576$ because reversing the sign of B only also reverses the sign of cosυ so that Bsinυcosυ is unchanged and also none of A′, C′ or F′ are changed.

Qn.14 (a) Solve $x^2 - 6x + 7 = 0$; fixed points are $3 \pm \sqrt{2}$ (approx. 4.414 and 1.586)
(b) $f' = 2x - 5$ modulus>1 in both cases, so both fixed points are repelling.
(c) $f(1) = 3$, $f(3) = 1$, hence (1,3) is a 2-cycle.
(d) $f'(1) = -3$ $f'(3) = 1$ |product|>1 so 2-cycle is repelling..
(e) $f(2) = 1$ so 2-cycle entered immediately.
(f) graph should show downward pointing parabola, with y-intercept = 7.
(g) (2,0)->(2,1)->(1,1), and the (1,3) 2-cycle is entered.

Qn.15 (a) $f'(x) = \dfrac{-ae^{a/x}}{x^2}$

(b) Apply Quotient Rule, derivative is

$$\frac{x^2\left(\dfrac{-ae^{a/x}}{x^2}\right) - 2xe^{a/x}}{x^4} = -\frac{(a+2x)e^{a/x}}{x^4}$$

(c) From (a)

$$\int \left(\frac{e^{a/x}}{x^2}\right) dx = -\frac{e^{a/x}}{a}$$

$$\int \frac{e^{a/x}}{x^3}\,dx = \int \left(\frac{e^{a/x}}{x^2}\right)\frac{1}{x}\,dx = -\frac{e^{a/x}}{ax} - \int \frac{e^{a/x}}{ax^2} dx$$

$$= -\frac{e^{a/x}}{ax} + \frac{e^{a/x}}{a^2} + c \quad = \frac{1}{a}\left(\frac{1}{a} - \frac{1}{x}\right)e^{a/x} + c$$

(d) substitute u=1/x so du=-dx/x^2.
Area=

$$\int_2^1 -e^u\,du = \left[e^u\right]_1^2 = e^2 - e$$

= 4.671 (to 4 sig figs)

(e) volume = $\pi \displaystyle\int_{1/2}^{1} \frac{e^{2/x}}{x^4} dx$

Again substitute u=1/x so du=-dx/x^2
Integral =

$$\pi \int_1^2 u^2 e^{2u}\,du$$

Integrate by parts twice :

$$\int_1^2 u^2e^{2u}du = \left[u^2e^{2u}\right]_1^2 - \int_1^2 ue^{2u}du$$

$$\int_1^2 ue^{2u}du = \left[ue^{2u}\right]_1^2 - \tfrac{1}{2}\int_1^2 e^{2u}du$$

$$\tfrac{1}{2}\int_1^2 e^{2u}du = \left[e^{2u}\right]_1^2 = \tfrac{1}{4}e^4 - \tfrac{1}{4}e^2$$

$$\int_1^2 u^2e^{2u}du = (2e^4 - \tfrac{1}{2}e^2) - (e^4 - \tfrac{1}{2}e^2) + (\tfrac{1}{4}e^4 - \tfrac{1}{4}e^2)$$

so volume $= \pi(1\tfrac{1}{4}e^4 - \tfrac{1}{4}e^2)$

= 208.6 (to 4 sig figs)

(Note : the above is an 'otherwise' solution. To follow the alternative route, integrate the result of part (b) to give :

$$\left(\frac{e^{2/x}}{x^2}\right) = \int -\frac{(2+2x)e^{2/x}}{x^4}dx = -2\int \frac{e^{2/x}}{x^4}dx - 2\int \frac{e^{2/x}}{x^3}dx$$

and from (c)

$$2\int \frac{e^{2/x}}{x^3}dx = \left(\frac{1}{2} - \frac{1}{x}\right)e^{2/x}$$

thus

$$\int \frac{e^{2/x}}{x^4}dx = -\frac{1}{2}\left(\frac{e^{2/x}}{x^2}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{x}\right)e^{2/x}$$

evaluated between the limits 0.5 and 1 which =

$$(2e^4 - \tfrac{3}{4}e^4) - (\tfrac{1}{2}e^2 - \tfrac{1}{4}e^2)$$

so volume $= \pi(1\tfrac{1}{4}e^4 - \tfrac{1}{4}e^2)$ =208.6 to 4 sig figs as before.)

Qn.16 (a)

·	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

(cf. also soln. 3.1 on D3 p.59)

(b) inverses are $1^{-1}=1$, $i^{-1}=-i$, $(-1)^{-1}=-1$, $(-i)^{-1}=i$

(c) B – neither 2 nor 4 have inverses.

(d) C – has 4 self inverse elements

(e) (i) and (ii) answers can be either

·	6	8	4	2
6	6	8	4	2
8	8	4	2	6
4	4	2	6	8
2	2	6	8	4

with isomorphism
(ω(1)=6, ω(i)=8, ω(-1)=4, ω(-i)=2)
or

·	6	2	4	8
6	6	2	4	8
2	2	4	8	6
4	4	8	6	2
8	8	6	2	4

with isomorphism is
(ω(1)=6, ω(i)=2, ω(-1)=4, ω(-i)=8)