MS221 – 2006 Solutions

Qn.1 (a) e^2=1+30/24 =54/24=9/4 so e=3/2.
Foci : ($\pm 3\sqrt{6}$,0) directrices $x = \pm 4/3\sqrt{6}$,

(b) Axes of symmetry : x=2, y= -3
Foci : ($2 \pm 3\sqrt{6}$,-3)
Directrices : $x = 2 \pm 4/3\sqrt{6}$

n.b. sketch should show the 'noses' of the hyperbola at approx. (7,-3) and (-3,-3), the asymptotes as approx. (y+3)=±1.1(x-2) and directrices at approx. x=5.3 and x= -1.3.

Qn.2 (a) $\frac{\sqrt{3}}{3}, \sqrt{5}$, $\tan(\phi - \theta) = \frac{\sqrt{5} - \frac{\sqrt{3}}{3}}{1 + \sqrt{15}/3}$

multiply top and bottom by $\sqrt{3}$:

$$\tan(\phi - \theta) = \frac{\sqrt{15} - 1}{\sqrt{3} + \sqrt{5}}$$

(b) multiply top and bottom by $\sqrt{5} - \sqrt{3}$, use $\sqrt{15} = \sqrt{5}\sqrt{3}$ and simplify to give

$$3\sqrt{3} - 2\sqrt{5}.$$

Qn.3 (a) 1/2 arctan(2/8) = 7.018°
(see p. 49 in Handbook)

(b) $\tan 2\theta = \frac{1}{4}$ => $\cos 2\theta = \frac{4}{\sqrt{17}}$

so $\cos^2\theta = \frac{1}{2}\left(1 + \frac{4}{\sqrt{17}}\right)$ $\sin^2\theta = \frac{1}{2}\left(1 - \frac{4}{\sqrt{17}}\right)$

$\sin^2\theta.\cos^2\theta = \frac{1}{17}.\frac{1}{4}$ $\sin\theta.\cos\theta = \frac{1}{2\sqrt{17}}$

$$A' = \frac{5}{2}\left(1 + \frac{4}{\sqrt{17}}\right) + \frac{1}{\sqrt{17}} - \frac{3}{2}\left(1 - \frac{4}{\sqrt{17}}\right) = 1 + \sqrt{17}$$

$$C' = \frac{5}{2}\left(1 - \frac{4}{\sqrt{17}}\right) - \frac{1}{\sqrt{17}} - \frac{3}{2}\left(1 + \frac{4}{\sqrt{17}}\right) = 1 - \sqrt{17}$$

Transformed conic *K* is
$5.12\, x^2 - 3.12\, y^2 + 60 = 0$

Qn.4 (a) $f'(x) = \frac{1}{4}x - \frac{3}{2}$
f '(x)<1 => x<10; *f '(x)*>-1 => x>2
so interval of attraction = (2,6)
(i) divergence (ii) convergence to x=4

Qn. 5 (a) $\begin{pmatrix} 2 & 4 \\ 0 & -3 \end{pmatrix}$
(b) (10,-6) , (8,-3)
(c) det = -6, answer = 9

Qn. 6 (a) eigenvalues are -1 and -2
eigenlines are y=1/2 x and y = 2/3 x.
(i) no - once a point is on one of the eigenlines, it remains on that eigenline.
(ii) yes - this is the behaviour associated with k=-1. (see p.63 in Handbook)

Qn.7 (a) $e^{-2x}\sec x(\tan x - 2)$
(b) $-\frac{1}{x\sqrt{x^2 - 1}}$

Qn.8 (a) $-3(x-1)\cos(\frac{1}{3}x) + 9\sin(\frac{1}{3}x) + c$
(b) substitute u=ln(x) :
$du = \frac{dx}{x}$, $I = \int \frac{du}{u^2} = -\frac{1}{\ln(x)} + c$

Qn.9 (a) $1 - x^2 + x^4 - x^6 + \ldots$
(b) -1<x<1
(c) $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$
(d) 0.30000 - 0.00900 + 0.00049 -0.00003
= 0.2915 to 4 dec. pl.

Qn.10 (a) 5, 4+3i, 11+i, 25 (b) 1/25(-1 + 18i)

Qn.11

·	3	6	9	12
3	9	3	12	6
6	3	6	9	12
9	12	9	6	3
12	6	12	3	9

(b) 6 is identity because
6.n = n.6 = n for all n in {3,6,9,12}

(c) 3^{-1}=12 6^{-1}=6 9^{-1}=9 12^{-1}=3

Qn.12 (a) 22 = 3.7 + 1 so 7.3= –1 (mod 22), 7.(-3)=1(mod 22) and therefore the inverse of 7 is 19.
(b) $E_7^{-1}(2) = 2^{19}$(mod 23). In modulo 23 arithmetic, 2^8=256=3, so 2^{16}=9, 2^{19}=72=3 (see Handbook p.84)

Qn.13 (a) 13, 35
(b) Solve r^2-5r+6=0, i.e. (r-3)(r-2)=0 to give r=3 and r = 2 as roots.
General solution is $u_n = A(3^n) + B(2^n)$
Use u_0 and u_1 : A + B = 2
3A + 2B = 5
Solution is A = B=1, so closed form is
$u_n = 3^n + 2^n$ (n=0,1,2,..)

(c) (5.35) - 13^2 = 6

(d) value of expression is 6^{n-1}.
(e) $u_{n-1}u_{n+1} = (3^{n-1} + 2^{n-1})(3^{n+1} + 2^{n+1})$
$= 3^{2n} + 2^{2n} + 3^{n-1}.2^{n-1}(9 + 4)$
$u_n^{\,2} = 3^{2n} + 2^{2n} + 2(3^n.2^n)$
difference is $3^{n-1}.2^{n-1}(13 - 12) = 6^{n-1}$

Qn.14 (a) Rule is : **x** |-> f^{-1}(**x**)

where $f^{-1} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$

image is (-x+3y, x-2y)

(ii) image of unit circle is

$(-x+3y)^2 + (x-2y)^2 = 1$, that is

$2x^2 - 10xy + 13y^2 = 1$

(b) (i) (0,0) -> (0,-2) (1,0) ->(0,-1)
(0,1)->(-1,-2)

(ii) substitute values *p*,*q*,*s*,*t*,*u*,*v* using Handbook p. 61):

g(x) = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

(iii) (h o g)**x** = h(g(**x**))

$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

= (identity matrix) times **x**

Qn.15 (a)

$$\int e^{ax}\cos(ax)dx = \frac{1}{a}e^{ax}\sin(ax) - \int e^{ax}\sin(ax)dx$$

$$\int e^{ax}\sin(ax)dx = -\frac{1}{a}e^{ax}\cos(ax) + \int e^{ax}\cos(ax)dx$$

by integration by parts. Substitute from the second line into the first and the result follows.

(b) Substitute u=ln(x) so $x=e^u$

du = 1/x dx, so dx = e^udu

$$I = \int e^u \cos u\,du = \frac{1}{2}e^u(\sin u + \cos u) + c$$

by part (a)

$$= \frac{1}{2}x(\sin(\ln x) + \cos(\ln x)) + c$$

(c) area = $\int_0^{\pi/2} e^x \cos x\,dx$

$= \frac{1}{2}\left[e^x(\sin x + \cos x)\right]_0^{\pi/2}$ by part (a)

$= \frac{1}{2}(e^{\pi/2} - 1)$

(d) volume =

$$\pi\int_0^{\pi/2} e^{2x}\cos^2 x\,dx = \frac{\pi}{2}\int_0^{\pi/2} e^{2x}(1+\cos 2x)dx$$

$$\frac{\pi}{2}\int_0^{\pi/2} e^{2x}dx + \frac{\pi}{8}\left[e^{2x}(\sin 2x + \cos 2x)\right]_0^{\pi/2}$$

by part (a)

$$= \frac{\pi}{4}(e^{\pi} - 1) + \frac{\pi}{8}(e^{\pi} - 1)$$

$$= \frac{\pi}{8}(e^{\pi} - 3)$$

Qn.16 (a) number ends in 0 so div by 10;
digit sum=45 so div by 3;
alt. digsum = 28-17=11 so div by 11;
hence number is divisible by 330.

(b) (i) n div by 5 and by 8 => n div by 20.
(ii) original is false, counter-example = 20.
(iii) div by 5 and 8=> div by 40 => div by 20.

(c) Add $\dfrac{1}{(r+2)(r+3)}$ to $\dfrac{r}{2(r+2)}$.

Denominator is 2(r+2)(r+3),
numerator is 2+r(r+3) = (r+1)(r+2)
Divide top and bottom by (r+2) to give the 'r+1' form.
1/(2.3) = 1/(2(1+2))
so the result is true for n=1. Therefore, by induction, it is true for all n.