MS221 – 2005 Solutions

Qn.1 (a) $t_{7,-3}$ (see p.48 in Handbook)
(b) (x-5, y+2) (c) $t_{2,-1}$: (x, y) |–> (x+2, y-1)

Qn.2 (a) curve is $y^2=20x$; parabola
(b) (i) e =1, (ii) (5,0) (iii) x= -5

Qn.3 (a) 19/5 (b) $u_n = 3+5(-\frac{2}{5})^n$
(c) tends to 3 in the long term
since $(-\frac{2}{5})^n \to 0$. (0<|-2/5|<1)

Qn.4 (a) Solve *f(x)=x* which is equivalent to solving $x^2 + 3x - 10=0$, i.e. (x+5)(x-2)=0, giving fixed points x=-5 and x = 2.
(b) *f'(x)* = 1/2 x + 7/4 so
f'(-5) = -3/4 (attracting)
f'(2) =11/4 (repelling)
(c) *f* does not have a 2-cycle. If it did then *fof* would have four intersections with y = x, two in common with *f*, and the other two at the points of the 2-cycle of *f*.

Qn.5 **D** = $\begin{pmatrix} (-3)^5 & 0 \\ 0 & 2^5 \end{pmatrix}$

$\mathbf{A^5} = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -243 & 0 \\ 0 & 32 \end{pmatrix}\begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} =$
$\begin{pmatrix} -2168 & 7700 \\ -550 & 1957 \end{pmatrix}$

Qn.6 (a) (i) (0,0), (1,0), (0,1), (1,1)
become (0,0), (0,1), (-1,-2), (-1,-1)
(b) (i) **B** = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
(ii) $\mathbf{A} = \mathbf{B^{-1}M} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$
(iii) An x-shear with parameter -2.

Qn.7 (a) 1 + 2x.arctan x
(b) -cotx

Qn. 8 (a) Integral= $\frac{2}{5}x^{5/2}\ln(4x) - \frac{2}{5}\int x^{5/2}\left(\frac{1}{x}\right)dx$

$= \frac{2}{5}x^{5/2}\ln(4x) - \frac{4}{25}x^{5/2} + c$

(b) u = exp(6 + cos(2x)) ; du = -2sin(2x)dx
Integral = $\int -\frac{1}{2}e^u du = -\frac{1}{2}\exp(6+\cos 2x) + c$

Qn. 9 (a) (i) $x - 1/2\, x^3 + 1/24\, x^5$
(ii) $1/3\, x^3 - 1/30\, x^5 + 1/840\, x^7$
(b) $x^2 - 1/6\, x^4 + 1/120\, x^6$
(c) xsin(x)

Qn.10 (a) modulus = 1, argument = $\frac{2\pi}{3}$
(b) $< 1, \frac{2\pi}{3} >$
(c) $\bar{w} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$
$w^2 = (\frac{1}{4} - \frac{3}{4}) - \frac{\sqrt{3}}{2}i = \bar{w}$
(d) $w^3 = 1$ because $3 \times \frac{2\pi}{3} = 2\pi$
(e) $< r, \theta + \frac{2\pi}{3} >$

Qn.11 43 = 2.18 +7 1 = 4 - 1.3
18 = 2.7 + 4 = 4 - (7 - 1.4)
7 = 1.4 + 3 = 2.4 - 1.7
4 = 1.3 + 1 = 2(18-2.7) - 1.7
then go to = 2.18 - 5.7
next col. -> = 2.18 - 5(43-2.18)
= -5.43 + 12.18
= 12.18 in mod 43
so inverse is 12.

Qn.12 (a) $c(n) \wedge d(n)$
(b) (i) 30 and 60 are simple examples.
(ii) $d(n) \Rightarrow (a(n) \wedge c(n))$
If n is divisible by 24 then n is divisible by 6 and is also divisible by 15.

Qn.13 See Handbook p.49
(a) $\frac{1}{2}\arctan\frac{-14\sqrt{2}}{-7} = \frac{1}{2}\arctan 2\sqrt{2}$
that is, $\tan 2\theta = 2\sqrt{2}$
(b) Simplify $\frac{2t}{1-t^2} = 2\sqrt{2}$ to form the quadratic $t^2\sqrt{2} + t - \sqrt{2} = 0$. Either factorise this as $(t\sqrt{2}-1)(t+\sqrt{2}) = 0$, so that the positive root is $1/\sqrt{2}$ and $\tan\theta = 1/\sqrt{2}$, or simply substitute the given answer.
(c) $\sin\theta = \sqrt{\frac{1}{3}}$ $\cos\theta = \sqrt{\frac{2}{3}}$
(d) $A' = 30\left(\frac{2}{3}\right) - 14\sqrt{2}\left(\frac{1}{\sqrt{3}}\right)\sqrt{\frac{2}{3}} + 37\left(\frac{1}{3}\right) = 23$

$C' = 30\left(\frac{1}{3}\right) + 14\sqrt{2}\left(\frac{1}{\sqrt{3}}\right)\sqrt{\frac{2}{3}} + 37\left(\frac{2}{3}\right) = 44$

Equation of K is $23x^2 + 44y^2 = 1012$.
Divide by 1012 : $\frac{x^2}{44} + \frac{y^2}{23} = 1$ which is the equation of an ellipse in standard form.
(e) $\arctan 1/\sqrt{2} = 35.26^{\circ}$
(f) $e = \sqrt{1 - \frac{23}{44}} == \sqrt{\frac{21}{44}} = \frac{1}{2}\sqrt{\frac{21}{11}}$
(g) eccentricity is invariant under rotation so answer is the same as (f).
(h) $y = x/\sqrt{2}$ $y = -x\sqrt{2}$

Qn.14 (a) (i) $\det\begin{pmatrix} 4-k & 3 \\ 6 & -3-k \end{pmatrix}=0$

=> $k^2 - k - 30 = 0$ $(k-6)(k+5)=0$

eigenvalues are 6 and -5

(ii) corresponding eigenlines are

2x - 3y = 0 (k=6) and 3x + y = 0 (k=-5).

eigenvectors are $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (k=6) and $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ (k=-5)

(iii) $\mathbf{P} = \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} 6 & 0 \\ 0 & -5 \end{pmatrix}$

(b) see Handbook p.63

(i) the initial point on eigenline y= -x corresponds to a -ve eigenvalue which means that the subsequent points should be on alternate halves of the eigenline.

(ii) since -4<-1 the points should be on alternate sides of the eigenline y = -1/7 x corresponding to k=2, which they are not.

(iii) since 2>1 the points should be on the same side of the eigenline y = -x corresponding to k= -4, which they are not.

Qn.15 (a)

(i) 1+sin x > 0 for all $x \neq (2n+1)\frac{\pi}{2}$

cos x > 0 for all x in domain

so *f(x)*>0 throughout domain.

(ii) $f'(x) = \frac{1+\sin x}{\cos^2 x}$ which is >0 throughout the domain so *f(x)* is an increasing function.

(b) $f(x) = \frac{(1+\sin x)(1-\sin x)}{\cos x(1-\sin x)}$

$= \frac{1-\sin^2 x}{\cos x(1-\sin x)} = \frac{\cos^2 x}{\cos x(1-\sin x)} = \frac{\cos x}{(1-\sin x)}$

If u(x) = 1- sin x, du= -cosx dx so integral is -du/u = -ln u, hence the answer.

(c) Area =

$\left[-(\ln(1-\sin x)\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \ln\left(1+\sqrt{3}/2\right) - \ln\left(1-\sqrt{3}/2\right)$

$= \ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right) = 2.634$ to 3 dp.

(d) (i) $f(x) = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \sec x + \tan x$

Volume of revolution =

$\pi\int_{-\pi/3}^{\pi/3} (\sec x + \tan x)^2 dx = \pi\int_{-\pi/3}^{\pi/3} (s^2 + t^2 + 2st)dx$

$= \pi\int_{-\pi/3}^{\pi/3} (2s^2 - 1 + 2st)dx$ using $t^2 = s^2 - 1$

Use the table of integrals on p.68 of the Handbook, and observe that *sec(x)tan(x)* is an odd function and hence its integral between limits symmetrical about the origin is zero. The value of the integral is thus

$\pi\int_{-\pi/3}^{\pi/3} (2\sec^2 x - 1)dx = \pi\left[2\tan x - x\right]_{-\pi/3}^{\pi/3}$

$= \pi\left(\left(2\sqrt{3} - \pi/3\right) - \left(-2\sqrt{3} + \pi/3\right)\right)$

$= \pi\left(4\sqrt{3} - 2\pi/3\right)$ = 15.186 to 3 dp.

Qn.16 (a) the set of real numbers except x=1

(b) *fof* = $\frac{\frac{x-1}{x} - 1}{\frac{x-1}{x}} = \frac{-1}{x-1}$ and so can be formed.

The result is equal to *g(x).*

(c) *fog=e*, *kok=e*, *kog=j*, *gok=h* (rules as given in the question).

(d) Inverses are paired (*e*,*e*), (*f*,*g*), (*h*,*h*), (*k*,*k*), (*j*,*j*)

(e) The group S is isomorphic to *G* because there are four self-inverse elements.