MS221/P

The Open University

Course Examination 2004

Exploring Mathematics

Tuesday 19 October 2004 10.00 am – 1.00 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I you should attempt as many questions as you can. Credit will be given for answers to no more than TWO questions in Part II. Your answers to each part should be written in the answer books provided.

In most questions, some marks will be awarded for intermediate steps in the working. A correct answer, not supported by working – as, for example, one taken from a calculator program – may not receive full credit.

72% of the available marks are assigned to Part I and 28% to Part II. In the examiners' opinion, most candidates would make best use of their time by finishing as much as they can of Part I before starting Part II.

You are advised not to cross through any work until you have replaced it with another solution to the same question. In Part II of the paper, if you submit attempts to more than two questions, your best two scores will count towards your result.

Graph paper is available from the invigilator, if you feel it would assist you in answering questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified**.

Put all your used answer books and your question paper together, with your signed desk record on top. Fix them all together with the fastener provided.

PART I

Instructions

(i) *You should attempt as many questions as you can in this part of the examination.*

(ii) *Part I carries 72% of the available examination marks. Each question carries an indication of the number of marks that are allocated to it.*

(iii) *You should record your answers to each question in the answer book(s) provided. You are strongly advised to show all your working, including any rough working.*

Question 1 – 5 marks

This question concerns the linear second-order recurrence sequence given by

$$u_0 = 1, \qquad u_1 = 2, \qquad u_{n+2} = 8u_{n+1} - 16u_n \qquad (n = 0, 1, 2, \ldots).$$

(a) Find the third term in the sequence. [*1*]

(b) Find a closed form for the sequence. [*4*]

Question 2 – 6 marks

This question concerns the curve with equation

$$x^2 - 4y^2 = 36.$$

(a) Show that this curve is a hyperbola in standard position, by giving its equation in standard form. [*1*]

(b) Find

(i) the eccentricity of the hyperbola;

(ii) the foci of the hyperbola;

(iii) the equations of the asymptotes of the hyperbola. [*3*]

(c) Sketch the hyperbola, showing its asymptotes and the points of intersection with the axes. [*2*]

Question 3 – 4 marks

(a) Write down, in the form $t_{p,q}$, the translation that moves the point $(-2, 3)$ to the point $(5, -2)$. [*1*]

(b) The translation $t_{4,-2}$ is applied to the output from the translation in part (a). Express the composite translation in the form

$$(x, y) \longmapsto (\qquad\qquad , \qquad\qquad).$$ [*1*]

(c) The line $y = 2x + 3$ is moved using the translation in part (a).

Write down the equation of the line after this translation. [*2*]

Question 4 – 3 marks

Suppose that $\tan A = \frac{1}{2}\sqrt{5}$ is the exact value of the tangent of an acute angle A.

(a) By considering an appropriate right-angled triangle, find the exact values of $\sin A$ and $\cos A$. [*2*]

(b) Use the values in part(a) and a trigonometric formula to find the exact value of $\sin(2A)$. [*1*]

Question 5 – 4 marks

(a) How many five-letter combinations can be chosen from nine different letters? [1]

(b) Use the Binomial Theorem to find the coefficient of x^4y^5 in the expansion of $(2x - 3y)^9$. [3]

Question 6 – 7 marks

Let f be the linear transformation that maps $(1, 0)$ to $(6, -8)$ and $(0, 1)$ to $(-3, 4)$, and let g be the linear transformation that maps $(1, 0)$ to $(1, 2)$ and $(0, 1)$ to $(-1, 2)$.

(a) Write down the matrices $\mathbf{A}$ and $\mathbf{B}$ that represent f and g, respectively. [1]

(b) Show that f is a flattening, and write down the equation of the line onto which f flattens the plane. [2]

(c) Find the matrix of the linear transformation $g \circ f$. [2]

(d) Describe the effect of $g \circ f$ on the plane. [2]

Question 7 – 4 marks

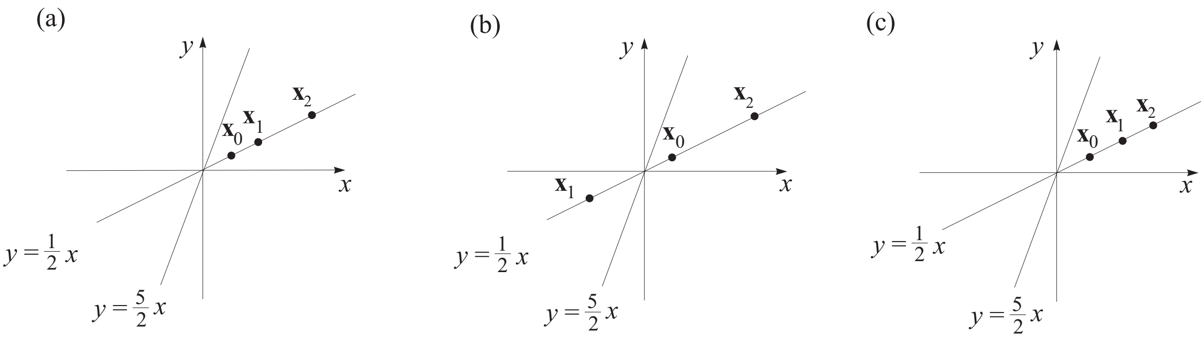
In this question, $\mathbf{A} = \begin{pmatrix} 6 & 3 \\ -2 & 1 \end{pmatrix}$.

(a) Find the eigenvalues of $\mathbf{A}$. [2]

(b) *Given that* the smaller and larger eigenvalues of $\mathbf{A}$ correspond to the eigenlines $y = -x$ and $y = -\frac{2}{3}x$, respectively, write down an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$. (You do not need to evaluate $\mathbf{P}^{-1}$.) [2]

Question 8 – 3 marks

The matrix $\mathbf{A}$ has eigenvalues -2 and 2, and corresponding eigenlines $y = \frac{5}{2}x$ and $y = \frac{1}{2}x$, respectively. Only one of the following three diagrams shows the first three points, $\mathbf{x}_0$, $\mathbf{x}_1$ and $\mathbf{x}_2$, of an iteration sequence with recurrence relation $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$. State which diagram is correct. For each of the other two diagrams, give a reason why the diagram cannot be correct.



[3]

Question 9 – 4 marks

Differentiate each of the following functions. (There is no need to simplify your answers.)

(a) $f(x) = \tan(\sqrt{x}) \quad (0 < x < \frac{1}{4}\pi^2)$ [*2*]

(b) $g(x) = \dfrac{e^{4x}}{\cos x} \quad (-\frac{1}{2}\pi < x < \frac{1}{2}\pi)$ [*2*]

Question 10 – 4 marks

This question concerns the function $f(x) = x + \ln x$.

(a) Show that the equation $f(x) = 0$ has a solution in the interval $(\frac{1}{2}, 1)$. [*1*]

(b) Show that, for this function f, the Newton-Raphson formula can be expressed as

$$x_{n+1} = \frac{x_n(1 - \ln x_n)}{x_n + 1}.$$ [*2*]

(c) Use this formula, with $x_0 = 0.6$, to calculate the value of x_1. Give your answer to three decimal places. [*1*]

Question 11 – 5 marks

(a) Using integration by parts, find the indefinite integral

$$\int (x+3)e^{x/2}\,dx.$$ [*3*]

(b) Using the substitution $u = \ln x$, or otherwise, find the indefinite integral

$$\int \frac{1}{x\,(1 + (\ln x)^2)}\,dx \quad (x > 0).$$ [*2*]

Question 12 – 5 marks

(a) Using a standard Taylor series, write down the first four non-zero terms of the Taylor series about 0 for the function

$$f(x) = (1 + x)^{-1/2}.$$ [*2*]

(b) Using your answer to part (a) and a further standard Taylor series, find the first two non-zero terms of the Taylor series about 0 for the function

$$g(x) = x(1 + x)^{-1/2} - \ln(1 + x).$$ [*2*]

(c) Use your result from part (b) to show that the corresponding Taylor polynomial estimate for $g(0.02)$ is 3.2×10^{-7} (to two significant figures). [*1*]

Question 13 – 5 marks

In this question, u and v are the complex numbers $u = 7 - 5i$ and $v = 3 + 2i$.

(a) Write down $|u|$ and $\overline{u}$. [*2*]

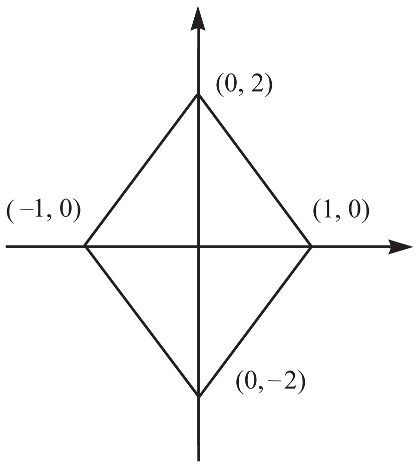
(b) Find a complex number z such that $uz = v$. [*3*]

Question 14 – 3 marks

(a) Find a number x in $\mathbb{Z}_{14}$ such that $x \times_{14} 11 = 1$. [*2*]

(b) Give an example of a non-zero number in $\mathbb{Z}_{14}$ that does not have a multiplicative inverse. [*1*]

Question 15 – 5 marks



(a) Write down the elements of the symmetry group of the rhombus shown above, using the notation r_θ and q_ϕ. [*2*]

(b) Compile a Cayley table for the group of symmetries in part (a). [*2*]

(c) Give an example of a group from the course that is isomorphic to the group in part (b), explaining your choice briefly. [*1*]

Question 16 – 5 marks

Prove by mathematical induction that

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \qquad [5]$$

for all positive integers n.

PART II

Instructions

(i) *Credit will be given for answers to no more than* **TWO** *questions from this part of the examination.*

(ii) *Each question in this part carries 14% of the total marks for the examination.*

(iii) *You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.*

(iv) *Show all your working.*

Question 17

(a) Show that the curve E given by the equation

$$4x^2 + 9y^2 + 40x - 18y + 73 = 0$$

is an ellipse, by showing that the equation can be written as

$$\frac{(x+5)^2}{9} + \frac{(y-1)^2}{4} = 1.$$ [*3*]

(b) Describe the translation that maps the ellipse F in standard position, $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$, onto E. [*1*]

(c) Find the coordinates of the foci of E, and write down the centre and the equations of the axes of symmetry of E. [*3*]

(d) Show that the directrices of E are $x = \frac{1}{5}(\pm 9\sqrt{5} - 25)$. [*2*]

(e) Sketch the ellipse E, and mark and identify on the sketch

(i) the foci and directrices;

(ii) the two axes of symmetry; [*4*]

(iii) the coordinates of points of intersection of E with the axes of symmetry.

(f) Write down parametric equations for E. [*1*]

Question 18

This question is about the function $f(x) = x^2 + \frac{7}{2}x$.

(a) (i) Find algebraically the fixed points of f. *[2]*

(ii) Classify each of the fixed points of f as attracting, repelling or indifferent, explaining your reasoning clearly. *[3]*

(b) (i) Express $f(f(x))$ as a polynomial of the form

$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$$

showing your working. *[2]*

(ii) *Given that* $f(f(x)) - x$ can be expressed as

$$(x^2 + \tfrac{5}{2}x)(x^2 + \tfrac{9}{2}x + \tfrac{9}{2}),$$

solve the equation $f(f(x)) = x$, showing your working. Deduce that the numbers -3 and $-\frac{3}{2}$ form a 2-cycle of f. *[3]*

(iii) Classify the 2-cycle of f as attracting, repelling or indifferent, explaining your reasoning clearly. *[2]*

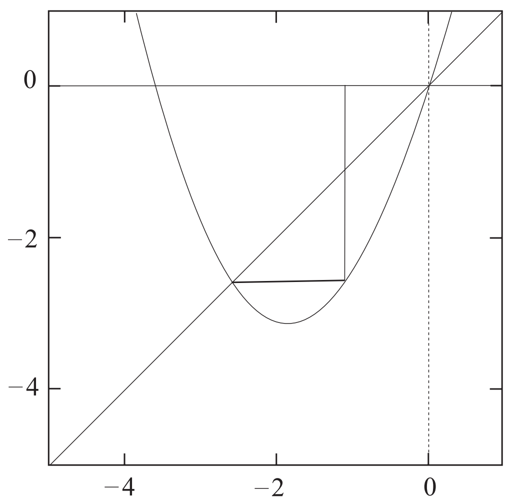
(c) Each of the following Mathcad diagrams illustrates the effect of iterating f with a particular initial term x_0.

In diagram (i), $x_0 = -1$ and the first 50 iterations are plotted.

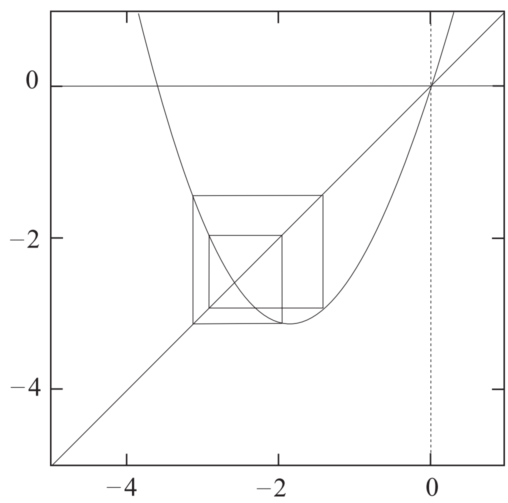
In diagram (ii), $x_0 = -\frac{3}{4}$ and the 51st to 100th iterations are plotted.

In each case, state what you think is the long-term behaviour of the iteration sequence. (You are not expected to read off particular values from diagram (ii).) *[2]*

(i)



(ii)



Question 19

(a) (i) Use integration by substitution to show that

$$\int \frac{x}{\sqrt{9+x^2}}\,dx = \sqrt{9+x^2} + c,$$

where c is an arbitrary constant. [3]

(ii) Hence find the area under the graph of $f(x) = x/\sqrt{9+x^2}$ from $x = 0$ to $x = 4$. [2]

(b) (i) Given that

$$\int x\sin(2x)\,dx = \tfrac{1}{4}\sin(2x) - \tfrac{1}{2}x\cos(2x) + b,$$

where b is an arbitrary constant, use a method of integration to show that

$$\int x^2\cos(2x)\,dx = \tfrac{1}{2}(x^2 - \tfrac{1}{2})\sin(2x) + \tfrac{1}{2}x\cos(2x) + c,$$

where c is an arbitrary constant. [4]

(ii) By using the result that you found in part (b)(i), find the volume of revolution obtained when the region under the graph of the function $g(x) = x\sin x$, from $x = 0$ to $x = \pi$, is rotated about the x-axis. Give your answer to two decimal places. [5]

Question 20

(a) Use Euclid's Algorithm to prove that 19 is its own inverse in $\mathbb{Z}_{30}$. [4]

A message, using the correspondence shown in the table below, is enciphered using the function

$f(m) \equiv m^{19} \pmod{31}$.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

(b) Explain why this is a correct form for an exponential cipher. [2]

(c) The ciphertext is

$\langle 28, 13, 12, 12, 5, 28, 28 \rangle$.

What was the message? [8]

[END OF QUESTION PAPER]